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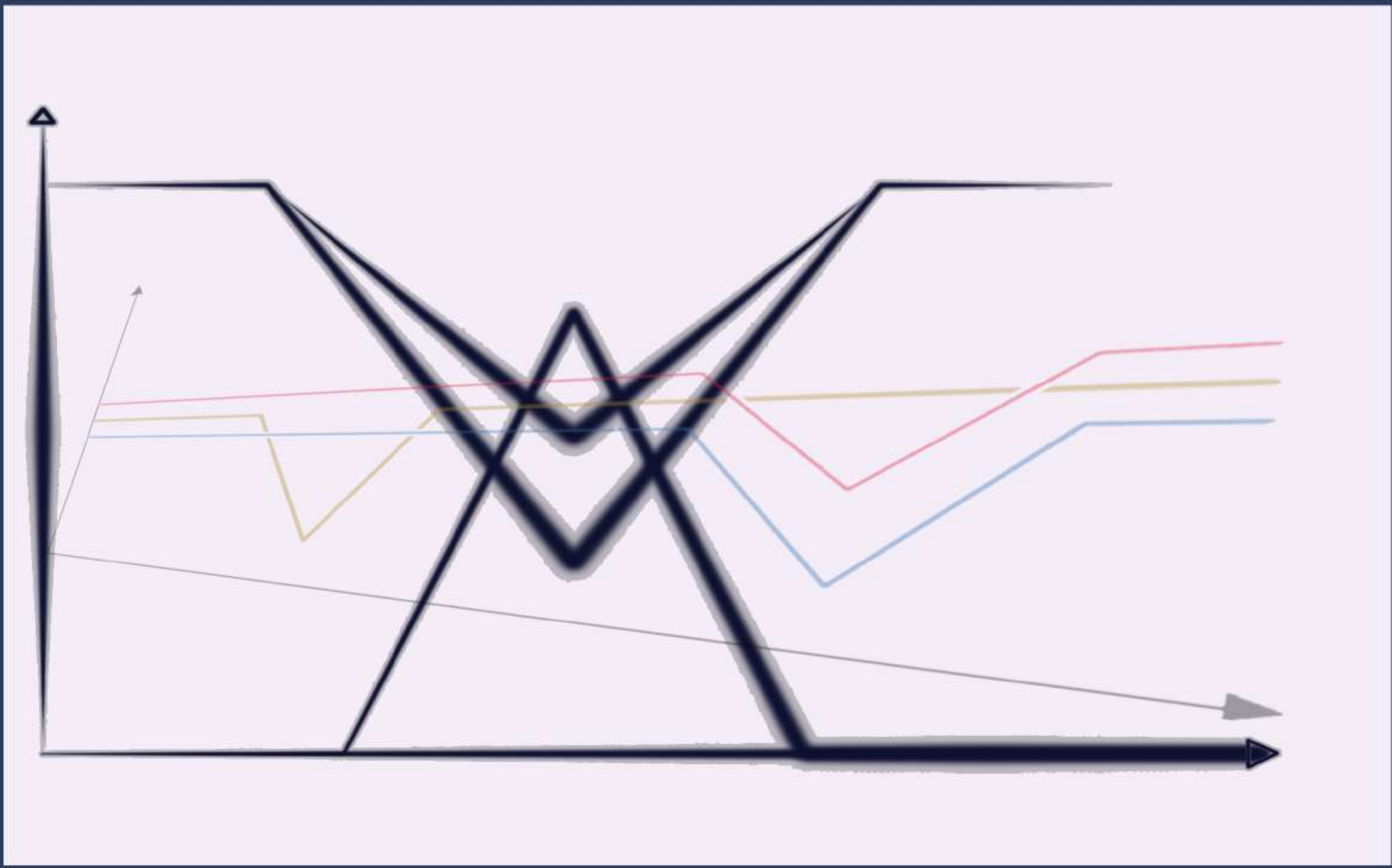

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Fuzzy Programming Approach to Bi-level Linear Programming Problems

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Abstract

In this study, we discussed a fuzzy programming approach to bi-level linear programming problems and their application. Bi-level linear programming is characterized as mathematical programming to solve decentralized problems with two decision-makers in the hierarchal organization. They become more important for the contemporary decentralized organization where each unit seeks to optimize its own objective. In addition to this, we have considered Bi-Level Linear Programming (BLPP) and applied the Fuzzy Mathematical Programming (FMP) approach to get the solution of the system. We have suggested the FMP method for the minimization of the objectives in terms of the linear membership functions. FMP is a supervised search procedure (supervised by the upper Decision Maker (DM)). The upper-level decision-maker provides the preferred values of decision variables under his control (to enable the lower level DM to search for his optimum in a wider feasible space) and the bounds of his objective function (to direct the lower level DM to search for his solutions in the right direction).

Keywords: Fuzzy set, Fuzzy function, Fuzzy linear programming, Bi Level programming.

1 | Introduction

Decision making problems in decentralized organizations are often modeled as stackelberg games, and they are formulated as bi-level mathematical programming problems. A bi-level problem with a single decision maker at the upper level and two or more decision makers at the lower level is referred to as a decentralized bi-level programming problem. Real-world applications under non cooperative situations are formulated by bi-level mathematical programming problems and their effectiveness is demonstrated.

The use of fuzzy set theory for decision problems with several conflicting objectives was first introduced by Zimmermann. Thereafter, various versions of Fuzzy Programming (FP) have been investigated and widely circulated in literature. The use of the concept of tolerance membership function of fuzzy set theory to Bi-Linear Programming Problems (BLPPs) for satisfactory decisions

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was first introduced by Lai in 1996 [1]. Shih and Lee further extended Lai's concept by introducing the compensatory fuzzy operator for solving BLPPs [2]. Sinha studied alternative BLP techniques based on Fuzzy Mathematical Programming (FMP).

The basic concept of these FMP approaches is the same as Fuzzy Goal Programming (FGP) approach which implies that the lower level DMs optimizes, his/her objective function, taking a goal or preference of the higher level DMs in to consideration. In the decision process, considering the membership functions of the fuzzy goals for the decision variables of the higher level DM, the lower level DM solves a FMP problem with a constraint on an overall satisfactory degree of the higher level DMs. If the proposed solution is not satisfactory, to the higher level DMs, the solution search is continued by redefining the elicited membership functions until a satisfactory solution is reached [2]. The main difficulty that arises with the FMP approach of Sinha is that there is possibility of rejecting the solution again and again by the higher level DMs and re-evaluation of the problem is repeatedly needed to reach the satisfactory decision, where the objectives of the DMs are over conflicting [2].

Taking in to account vagueness of judgments of the decision makers, we will present interactive fuzzy programming for bi-level linear programming problems. In the interactive method, after determining the fuzzy goals of the decision makers at both levels, a satisfactory solution is derived by updating some reference points with respect to the satisfactory level. In the real world, we often encounter situations where there are two or more decision makers in an organization with a hierarchical structure, and they make decisions in turn or at the same time so as to optimize their objective functions. In particular, consider a case where there are two decision makers; one of the decision makers first makes a decision. Such a situation is formulated as a bi-level programming problem. Although a large number of algorithms for obtaining stackelberg solutions have been developed, it is also known that solving the mathematical programming problems for obtaining stackelberg solution is NP-hard [3]. From such difficulties, a new solution concept which is easy to compute and reflects structure of bi-level programming problems had been expected [4] proposed a solution method, which is different from the concept of stackelberg solutions, for bi-level linear programming problems with cooperative relationship between decision makers. Sakawa and Nishizaki [5] present interactive fuzzy programming for bi-level linear programming problems. In order to overcome the problem in the methods of [4], after eliminating the fuzzy goals for decision variables, they formulate the bi-level linear programming problem.

In their interactive method, after determining the fuzzy goals of the decision makers at all the levels, a satisfactory solution is derived efficiently by updating the satisfactory degree of the decision maker at the upper level with considerations of overall satisfactory balance among all the levels. By eliminating the fuzzy goals for the decision variables to avoid such problems in the method of [4]-[6] develop interactive fuzzy programming for bi-level linear programming problems. Moreover, from the viewpoint of experts' imprecise or fuzzy understanding of the nature of parameters in a problem-formulation process, they extend it to interactive fuzzy programming for bi-level linear programming problems with fuzzy parameters [5]. Interactive fuzzy programming can also be extended so as to manage decentralized bi-level linear programming problems by taking in to consideration individual satisfactory balance between the upper level DM and each of the lower level DMs as well as overall satisfactory balance between the two levels [7]. Moreover, by using some decomposition methods which take advantage of the structural features of the decentralized bi-level problems, efficient methods for computing satisfactory solutions are also developed [7] and [8].

Recently, [9]-[11] considered the L-R fuzzy numbers and the lexicography method in conjunction with crisp linear programming and designed a new model for solving FFLP. The proposed scheme presented promising results from the aspects of performance and computing efficiency. Moreover, comparison between the new model and two mentioned methods for the studied problem shows a remarkable agreement and reveals that the new model is more reliable in the point of view of optimality. Also, an author in [12]-[15] has been proposed a new efficient method for FFLP, in order to obtain the fuzzy optimal solution with unrestricted variables and parameters. This proposed method is based on crisp nonlinear programming and has a simple structure.

Furthermore, several authors deal with the modeling and optimization of a bi-level multi-objective production planning problem, where some of the coefficients of objective functions and parameters of constraints are multi-choice. They has been used a general transformation technique based on a binary variable to transform the multi-choices parameters of the problem into their equivalent deterministic form [16]-[21].

In this study, we discuss a procedure for solving bi-level linear programming problems through linear FMP approach. In order to reach the optimal solution of bi-level linear programming problems, using fuzzy programming approach, the report contains section three chapters. In section two we describe the basic concept of fuzzy set, and linear programming using fuzzy approach. In section three the basic concept of bi level linear programming characteristics and general model of mathematical formulation of bi -level linear programming problems are presented. In section, four the procedure for solving bi-level linear programming problems and FMP solution approach are discussed.

2| Preliminary

2.1| Fuzzy Set Theory

Fuzzy set theory has been developed to solve problems where the descriptions of activities and observations are imprecise, vague, or uncertain. The term “fuzzy” refers to a situation where there are no well-defined boundaries of the set of activities or observations to which the descriptions apply. For example, one can easily assign a person 180 cm tall to the class of tall men”. But it would be difficult to justify the inclusion or exclusion of a 173 cm tall person to that class, because the term “tall” does not constitute a well- defined boundary. This notion of fuzziness exists almost everywhere in our daily life, such as a ”class of red flowers,” a “class of good shooters,” a “class of comfortable speeds for travelling,” a “number close to 10,”etc.These classes of objects cannot be well represented by classical set theory. In classical set theory, an object is either in a set or not in a set. An object cannot partially belong to a set .In fuzzy set theory, we extend the image set of the characteristic function from the binary set $B = \{0 ,1\}$ which contains only two alternatives, to the unit interval $U = [0,1]$ which has an infinite number of alternatives. We even give the characteristic function a new name, the membership function, and a new symbol μ, instead of χ. The vagueness of language, and its mathematical representation and processing, is one of the major areas of study in fuzzy set theory.

2.2| Definition of Fuzzy and Crisp Sets

Definition 1. Let X be a space of points (objects) called universal or referential set .An ordinary (a crisp) subset A in X is characterized by its characteristic function X_A as mapping from the elements of X to the elements of the set {0,1} defined by;

$$X_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}.$$

Where $\{0, 1\}$ is called a valuation set. However, in the fuzzy set t, the membership function will have not only 0 and 1 but also any number in between. This implies that if the valuation set is allowed to be the real interval $[0, 1]$, A is called a fuzzy set.

Definition 2. If X is a collection of objects denoted by x, then a fuzzy set A is a set of ordered pairs denoted by $A = \{(x, \mu_A(x)) \mid x \in X\}$. Where $\mu_A(x): X \to [0,1]$ is called membership function or degree of membership (degree of compatibility or degree of truth).

Definition 3. A fuzzy set A in a non empty set X is categorized by its membership function $\mu_A(x)\colon X \rightarrow [0,1]$ and $\mu_A(x)$ is called the degree of membership of element x in fuzzy set A for each x is an element of X that makes values in the interval $[0,1]$.

Definition 4. Let X be a universal set and A is a subset of X. A fuzzy set of A in x is a set of ordered pairs $A = \{(x, \mu_A(x)) \mid x \in X\}$ where, $\mu_A(x) \rightarrow [0,1]$ is called the membership function at x in membership, the value one is used to represent complete membership and value zero is used to represent intermediate degree of membership.

Example 1. let $X = \{a, b, c\}$ and define the fuzzy set A as follows:

$$\mu_A(a) = 1.0, \qquad \mu_A(b) = 0.7, \qquad \mu_A(c) = 0.4,$$

$$A = \{(a, 1.0), (b, 0.7), (c, 0.4)\}.$$

Note. The statement, $\mu_A(b) = 0.7$ is interpreted as saying that the membership grade of 'b' in the fuzzy set A is seven-tenths. i.e. the degree or grade to which b belongs to A is 0.7.

Definition 5. A fuzzy set $A = \varnothing$ if and only if it is identically zero on X.

Definition 6. If two fuzzy sets A andfuzzy set B are equal then $A = B$, if and only if $A(x) = B(x), \ \forall x \in X$.

2.3| Fuzzy Linear Programming

Crisp linear programming is one of the most important operational research techniques. It is a problem of maximizing or minimizing a crisp objective function subject to crisp constraints (crisp linear-inequalities and/or equations). It has been applied to solve many real world problems but it fails to deal with imprecise data, that is, in many practical situations it may not be possible for the decision maker to specify the objective and/or the constraint in crisp manner rather he/she may have put them in "fuzzy sense". So many researchers succeeded in capturing such vague and imprecise information by fuzzy programming problem. In this case, the type of the problem he/she put in the fuzziness should be specified, that means, there is no general or unique definition of fuzzy linear problems. The fuzziness may appear in a linear programming problem in several ways such as the inequality may be fuzzy (p1–FLP), the objective function may be fuzzy (P2-FLP) or the parameters c, A, b may be fuzzy (P3-FLP) and so on.

Definition 7. If an imprecise aspiration level is assigned to the objective function, then this fuzzy objective is termed as fuzzy goal. It is characterized by its associated membership function by defining the tolerance limits for achievement of its aspired level.

We consider the general model of a linear programming

$$\begin{aligned} &\max \ C^T x, \\ &\text{s.t.} \\ &A_i x \le b_i \quad (i = 1,2,3, \dots m), \\ &x \ge 0, \end{aligned} \tag{1}$$

Where A_i is an n-vector C is an n-column vector and $x \in \mathbb{R}^n$.

To a standard linear programming *Problem (1)* above, taking in to account the imprecision or fuzziness of a decision maker's judgment, Zimmermann considers the following linear programming problem with a fuzzy goal (objective function) and fuzzy constraints.

$$C^T x \lesssim Z_0, \quad (1a)$$

$$A_i x \lesssim b_i \quad (i = 1,2,3, \ldots m), \quad (1b) \qquad (2)$$

$$x \geq 0.$$

Where the symbol $\lesssim$ denotes a relaxed or fuzzy version of the ordinary inequality $<$. From the decision maker's preference, the fuzzy goal (1a) and the fuzzy constraints (1b) mean that the objective *function* $C^T x$ should be "essentially smaller than or equal to" a certain level Z_0, and that the values of the constraints AX should be "essentially smaller than or equal to" b, respectively. Assuming that the fuzzy goal and the fuzzy constraints are equally important, he employed the following unified formulation.

$$Bx \lesssim b',$$

$$x \geq 0.$$

Where $B = \begin{bmatrix} C \\ A_i \end{bmatrix}$ *and* $b' = \begin{bmatrix} Z_0 \\ b_i \end{bmatrix}$.

Definition 8. Fuzzy decision is the fuzzy set of alternatives resulting from the intersection of the fuzzy constraints and fuzzy objective functions. Fuzzy objective functions and fuzzy constraints are characterized by their membership functions.

2.4 | Solution Techniques of Solving Some Fuzzy Linear Programming Problems

The solution techniques for fuzzy linear programming problems follow the following procedure. We consider the following linear programming problem with fuzzy goal and fuzzy constraints (the coefficients of the constraints are fuzzy numbers).

Where $\widetilde{a_{ij}}$ and $\widetilde{b_i}$ are fuzzy numbers with the following linear membership functions:

$$\mu_{ij}(x) = \begin{cases} 1, & \text{if } x \leq a_{ij}, \\ \dfrac{a_{ij} + d_{ij} - x}{d_{ij}}, & \text{if } a_{ij} < x < a_{ij} + d_{ij}, \\ 0, & \text{if } x \geq a_{ij} + d_{ij}. \end{cases}$$

$$\mu_{\widetilde{b_i}}(x) = \begin{cases} 1, & \text{if } x \leq b_i, \\ \dfrac{b_i + p_i - x}{p}, & \text{if } b_i < x < b_i + p_i, \\ 0, & \text{if } x \geq b_i + p_i. \end{cases}$$

and $x \in R, dij > 0$ is the maximum tolerance for the corresponding constraint coefficients and p_i is the maximum tolerance for the i^{th} constraint. For defuzzification of the problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values. These optimal values z_l and z_u can be defined by solving the following standard linear programming problems, for which we assume that both of them have finite optimal values.

$$z_1 = \max \sum_{j=1}^{n} c_j x_j,$$
$$\text{s.t.}$$
$$\sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \le b_i,\ \ 1 \le i \le m)\ x_j \ge 0,$$
$$\text{and} \tag{3}$$
$$z_2 = \max \sum_{j=1}^{n} c_j x_j\ ,$$
$$\text{s.t.}$$
$$\sum_{j=1}^{n} a_{ij} x_j \le b_i + p_i,\ \ 1 \le i \le m) x_j \ge 0.$$

Let $z_l = min(z_1, z_2)$ and $z_u = max(z_1, z_2)$. The objective function takes values between z_l and z_u while the constraint coefficients take values between a_{ij} and $a_{ij} + d_{ij}$and the right-hand side numbers take values between b_i and $b_i + p_i$ .Then, the fuzzy set optimal values, G, which is a subset of R^n is defined by:

$$\mu_G(x) = \begin{cases} 0, & \text{if } \sum_{j=1}^{n} c_j x_j \le z_l, \\ \dfrac{\sum_{j=1}^{n} c_j x_j - z_l}{z_u - z_l}, & \text{if } z_l < \sum_{j=1}^{n} c_j x_j \le z_u. \\ 1, & \text{if } \sum_{j=1}^{n} c_j x_j \ge z_u. \end{cases}$$

The fuzzy set of the i^{th} constraint, C_i , which is a subset of R^n is defined by:

$$\mu_{ci}(x) = \begin{cases} 0, & \text{if } b_i \le \sum_{j=1}^{n} a_{ij} x_j\ , \\ \dfrac{b_i - \sum_{j=1}^{n} a_{ij} x_j}{\sum_{j=1}^{n} d_{ij} x_j + p_i}, & \text{if } \sum_{j=1}^{n} a_{ij} x_j < b_i < \sum_{j=1}^{n} \left(a_{ij} x_j + d_{ij}\right) x_j + p_i. \\ 1, & \text{if } b \ge \sum_{j=1}^{n} \left(a_{ij} x_j + d_{ij}\right) x_j + p_i. \end{cases}$$

Using the above membership functions $\mu_{ci}(x)$ and $\mu_G(x)$and following Bellmann and Zadeh approach, we construct the membership function $\mu_D(x)$as follows: $\mu_D(x) = \text{mini}(\mu_G(x), \mu_{ci}(x))$.

Where $\mu_D(x)$ is the membership function of the fuzzy decision set. The min. section is selected as the aggregation operator. Then the optimal decision x^* is the solution of $x^* = \arg(\max \text{mini}\{\mu_G(x), \mu_{ci}(x)\}$.

Then, *Problem (1)* is reduced to the following crisp problem by introducing the auxiliary variable λ which indicates the common degree of satisfaction of both the fuzzy constraints and objective function.

$$\max \lambda,$$

s. t.

$$\mu_G(x) \geq \lambda,$$

$$\mu_{ci}(x) \geq \lambda,$$

$$x \geq 0, 0 \leq \lambda \leq 1,\ 1 \leq i \leq m.$$

This problem is equivalent to the following non-convex optimization problem

$$\max \lambda,$$

$$\lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j - z_1 \leq 0,$$

$$\sum_{j=1}^{n} \left(a_{ij} + \lambda d_{ij}\right) x_j + \lambda p_i - b_i \leq 0\,,$$

$$x \geq 0, 0 \leq \lambda \leq 1, \qquad 1 \leq i \leq m\,.$$

Which contains the cross product terms λx_j that makes non- convex. Therefore, the solution of this problem requires the special approach such as fuzzy decisive method adopted for solving general non-convex optimization problems. Here solving the above linear programming problem gives us an optimum $\lambda^* \in [0,1]$. Then the solution of the problem is any $x \geq 0$ satisfying the problem constraint with $\lambda = \lambda^*$.

3| Bi-Level Programming

3.1| Basic Definitions

3.1.1| Decision making

Decision making is a process of choosing an action (solution) from a set of possible actions to optimize a given objective.

3.1.2| Decision making under multi objectives

In most real situation a decision maker needs to choose an action to optimize more than one objective simultaneously. Most of these objectives are usually conflicting. For example, a manufacturer wants to increase his profit and at the same time want to produce a product with better quality. Mathematically a multi objective optimization with k objectives, for a natural number $K > 1$, can be given as:

$$\max F(x) = (f_1(x), \qquad f_2(x), \dots, f_k(x))\,,$$

s. t.

$$x \in S \subseteq \mathbb{R}^n.$$

3.1.3| Hierarchical decision making

An optimization problem which has other optimization problems in the constraint set and has a decision maker for each objective function controlling part of the variables is called multi-level optimization problem. If there are only two nested objective functions then it is called a bi-level optimization problem. The decision maker at the first level, with objective function f_1, is called the leader and the other decision makers are called the followers. A solution is supposed to fulfill all the

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feasibility conditions and optimize each objectives it is uncommon to find a solution which makes all the decision makers happy. Hence to choose an action the preference of the decision makers for all the levels or objectives play a big role.

3.1.4| Bi-level programming (BLP)

is a mathematical programming problem that solves decentralized planning problems with two DMs in a two level or hierarchical organization. It has been studied extensively since the 1980s. It often represents an adequate tool for modeling non-cooperative hierarchical decision process, where one player optimizes over a subset of decision variables, while taking in to account the independent reaction of the other player to his or course of action. In the real world, we often encounter situations where there are two or more decision makers in an organization with a hierarchical structure, and they make decisions in turn or at the same time so as to optimize their objective functions. In particular, consider a case where there are two decision makers; one of the decision makers first makes a decision, and then the other who knows the decision of the opponent makes a decision. Such a situation is formulated as a bi-level programming problem. We call the decision maker who first makes a decision the leader, and the other decision maker the follower. For bi-level programming problems, the leader first specifies (decides) a decision and then the follower determines a decision so as to optimize the objective function of the follower with full knowledge of the decision of the leader. According to this rule, the leader also makes a decision so as to optimize the objective function of self. This decision making process is extremely practical to such decentralized systems as agriculture, government policy, economic systems, finance, warfare, transportation, network designs, and is especially for conflict resolution.

Bi-level programming is particularly appropriate for problems with the following characteristics:

- ***Interaction:*** *Interactive decision-making units within a predominantly hierarchical structure.*
- ***Hierarchy:*** *Execution of decision is sequential, from upper to lower level.*
- ***Full information:*** *Each DM is fully informed about all prior choices when it is his turn to move.*
- ***Nonzero sum:*** *The loss for the cost of one level is unequal to the gain for the cost of the other level. External effect on a DM's problem can be reflected in both the objective function and the set of feasible decision space.*
- *Each DM controls only a subset of the decision variables in an organization.*

3.2| Mathematical Formulation of a Bi-Level Linear Programming Problem (BLPP)

For the bi-level programming problems, the leader first specifies a decision and then the follower determines a decision so as to optimize the objective function of self with full knowledge of the decision of the leader. According to this rule, the leader also makes a decision so as to optimize the objective function of self. The solution defined as the above mentioned procedure is a stackelberg solution.

A bi-level LPP for obtaining the stackelberg solution is formulated as:

$$\begin{aligned}
&\max_{x_1.} z_1(x_1, x_2) = c_1x_1 + d_1x_2,\\
&\text{Where } x_2 \text{ solves}\\
&\max_{x_2,} z_2(x_1, x_2) = c_2x_1 + d_2x_2,\\
&\text{s.t.}\\
&Ax_1 + Bx_2 \leqq b.
\end{aligned} \tag{4}$$

Where $c_i, i = 1,2$ are n_1-dimensional row coefficient vector$d_i, i = 1,2$, are n_2- dimensional row coefficient vector, A is an mxn1 coefficient matrix, B is a mxn_2 coefficient matrix, b-is an m-dimensional column constant vector. In the bi-level linear programming problem above$z_1(x_1, x_2)$, and $z_2(x_1, x_2)$represent the objective functions of the leader and the follower, respectively, and x_1 and x_2 represent the decision variables of the leader and the follower respectively. Each decision maker knows the objective function of self and the constraints. The leader first makes a decision, and then the follower makes a decision so as to maximize the objective function with full knowledge of the decision of the leader. Namely, after the leader chooses x_1, he solves the following linear programming problem:

$$\max_{x_2} z_2(x_1, x_2) = c_2x_1 + d_2x_2, \tag{5}$$

$$\text{s.t.}$$

$$Bx_2 \leq b - Ax_1,$$

$$x_2 \geqq 0.$$

And chooses an optimal solution $x_2(x_1)$ to the problem above as a rational response. Assuming that the follower chooses the rational response, the leader also makes a decision such that the objective function $z_1(x_1, x_2(x_1))$ is maximized.

3.3 | BLP Problem Description

The linear bi-level programming problem is similar to standard linear programming, except that the constraint region is modified to include a linear objective function constrained to be optimal with respect to one set of variables. The linear BLPP characterized by two planners at different hierarchical levels each independently controlling only a set of decision variables, and with different conflicting objectives. The lower- level executes its policies after and in view of, the decision of the higher level , and the higher level optimizes its objective independently which is usually affected by the reactions of the lower level. Let the control over all real-valued decision variables in the vector $x = (x_1^1, x_1^2, \dots, x_1^{N(1)}, x_2^1, x_2^2, \dots, x_2^{N(2)})$ be partitioned between two planners ,hereafter known as level-one(the superior or top planner) and level-two(the inferior or bottom planner).Assume that the level-one has control over the vector $x = (x_1^1, x_1^2, \dots, x_1^{N(1)})$, the first $N(1)$ components of the vector x, and that the level-two has control over the vector $x = (x_2^1, x_2^2, \dots, x_2^{N(2)})$ the remaining $N(2)$ components .Further, assume that $f_1, f_2: R^{N(1)} x\, R^{N(2)} \rightarrow R^1$ linear. Then, the linear BLPP can be formulated as:

$$\max_{x_1.} f_1(x_1, x_2) = c_1x_1 + d_1x_2, \tag{6}$$

$$\text{Where } x_2 \text{ solves}$$

$$\max_{x_2,} f_2(x_1, x_2) = c_2x_1 + d_2x_2, \tag{7}$$

$$\text{s.t.}\ (x_1, x_2) \in S.$$

Where $S \subseteq R^{N(1)+N(2)}$ is the feasible choices of (x_1, x_2), and is closed and bounded. For any fixed choice of x_1 , level-two will choose a value of x_2 to maximize the objective function $f_1(x_1, x_2)$. Hence, for each feasible value of x_1, level-two will react with a corresponding value of x_2. This induces a functional reaction ship between the decisions of level-one and the reactions of level-two. Say, $x_2 = W(x_1)$ .We will assume that the reaction function, $W(.)$, is completely known by level one.

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Definition 9. The set $Wf_2(S)$ given by $Wf_2(S) = \{(x_1^*, x_2^*) \in S: f_2(x_1^*, x_2^*) = max\ f_2(x_1^*, x_2^*)$ is the set of rational reactions of $f_2$over S. Hence level-one is really restricted to choosing a point in the set of rational reactions of f_2 over S. So, if level-one wishes to maximize its objective function, $f_1(x_1, x_2)$,by controlling only the vector x_1, it must solve the following mathematical programming problem:

$$\begin{aligned}&\max f_1(x_1, x_2),\\&\text{s. t.}\\&(x_1, x_2) \in Wf_2(S).\end{aligned} \tag{8}$$

For convenience of notation and terminology, we will refer to $S^1 = Wf_2(S)$ as the level-one feasible region or in general, the feasible region, and $S^1 = S$ as the level two feasible regions.

The following are the basic concepts of the bi-level linear programming problem of *Eq. 3*:

The feasible region of the bi-level linear programming problem: $S = \{(x_1, x_2): Ax_1 + Bx_2 \leqq b\}$.

The decision space (feasible set) of the follower after x_1 is specified by the leader: $S(x_1) = \{x_2 \geqq 0: Bx_2 < b - Ax_1, x_1 \geqq 0\}$.

The decision space of the leader: $S_x = \{x_1 \geqq 0$ there is an x_2 such that $Ax_1 + Bx_2 \leqq b, x_2 \geqq 0\}$.

The set of rational responses of the follower for x_1 specified by the leader

$$R(x_1) = \begin{cases} x_2 \geqq 0: x_2 \in \arg\max z_1(x_1, x_2) \\ \qquad\qquad x_2 \in S(x_1) \end{cases}.$$

Inducible region: $IR = \{(x_1, x_2): (x_1, x_2) \in S, x_2 \in R(x_1)\}$.

Stackleberg solution: $\{(x_1, x_2): (x_1, x_2) \in \arg\max z_1(x_1, x_2), (x_1, x_2) \in R(x_1)\}$.

Computational methods for obtaining stackelberg solution to bi-level linear programming problems are classified roughly in to three categories. These are

The vertex enumeration approach [2]. This takes advantage of the property that there exists a stackelberg solution in a set of extreme points of the feasible region. The solution search procedure of the method starts from the first best point namely an optimal solution to the upper level problem which is the first best solution, is computed, and then it is verified whether the first best solution is also an optimal solution to the lower level problem. If the first best point is not the stackelberg solution, the procedure continues to examine the second best solution to the problem of the upper level, and so forth.

The Kuhn-Tucker approach. In this approach, the leader's problem with constraints involving the optimality conditions of the follower's problem is solved.

The penalty function approach. In this approach, a penalty term is appended to the objective function of the leader so as to satisfy the optimality of the follower's problem.

Fuzzy approach:-that will be discussed in detail under the next chapter.

4| Fuzzy Approach to Bi-Level Linear Programming Problems

4.1| Fuzzy Bi-Level Linear Programming

As discussed under chapter two, a bi-level linear programming problem is formulated as:

$$\max f_1(x_1, x_2) = c_{11}x_1 + c_{12}x_2,$$

$$x_1. \tag{9}$$

Where x_2 solves

$$\max f_2(x_1, x_2) = c_{21}x_1 + c_{22}x_2,$$

s. t.

$$A_1x_1 + A_2x_2 \leq b,$$

$$(x_1, x_2) \geq 0.$$

Where $x_i, i = 1,2$ is an n_i-dimensional decision variable column vector ;

$C_{i1}, i = 1,2$ is an n_1-dimensional constant column vector;

$C_{i2}, i = 1,2$ is an n_2-dimensional constant column vector;

b-is an m-dimensional constant column vector, and

$A_i, i = 1,2$ is an mxni coefficient matrix.

For the sake of simplicity, we use the following notations:

$X = (x_1, x_2) \in R^{n_1+n_2}, C_i = (C_{i1}, C_{i2}), i = 1,2$ and $A = [A_1, A_2]$ and Let DM_1 denotes the decision maker at the upper level and DM_2 denotes the decision maker at the lower level. In the bi-level linear programming problem (7) above, $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$represent the objective functions of DM_1 and DM_2 respectively; and x_1 and x_2 represent the decision variables of DM_1 and DM_2 respectively.

Instead of searching through vertices as the k^{th} best algorithm, or the transformation approach based on Kuhn-Tucker conditions, we here introduce a supervised search procedure (supervised by DM_1) which will generate (non dominated) satisfactory solution for a bi-level programming problem. In this solution search, DM_1 specifies(decides) a fuzzy goal and a minimal satisfactory level for his objective function and decision vector and evaluates a solution proposed by DM_2, and DM_2 solves an optimization problem, referring to the fuzzy goal and the minimal satisfactory level of DM_1. The DM_2 then presents his/her solution to the DM_1. If the DM_1 agrees to the proposed solution, a solution is reached and it is called a satisfactory solution here. If he/she rejects this proposal, then DM_1 will need to re-evaluate and change former goals and decisions as well as their corresponding leeway or tolerances until a satisfactory solution is reached. It is natural that decision makers have fuzzy goals for their objective functions and their decision variables when they take fuzziness of human judgments in to consideration .For each of the objective functions $f_i(x)$, $i = 1,2$, assume that the decision makers have fuzzy goals such as "the objective function $f_i(x)$ should be substantially less than or equal to some value q_i " and the range of the decision on x_i , $i = 1,2$,should be " around $x_i^*$with its negative and positive –side tolerances $p_i^-$and p_i^+,respectively.

We obtain optimal solution of each DM_1 and DM_2 calculated in isolation. If the individual optimal solution x_i^0, $i =$ 1.2; are the same then a satisfactory solution of the system has been attained. But this rarely happens due to conflicting objective functions of the two DMs. The decision-making process then begins at the first level. Thus, the first-level DM provides his preferred ranges for f_1 and decision vector x_1 to the second level DM. This information can be modeled by fuzzy set theory using membership functions.

4.2| Fuzzy Programming Formulation of BLPPs

To formulate the fuzzy programming model of a BLPP, the objective functions f_i, $(i = 1,2)$ and the decision vectors x_i, $(i = 1,2)$ would be transformed in to fuzz goals by means of assigning an aspiration level (the optimal solutions of both of the DMs calculated in isolation can be taken as the aspiration levels of their associated fuzzy goals) to each of them. Then, they are to be characterized by the associated membership functions by defining tolerance limits for achievement of the aspired levels of the corresponding fuzzy goals.

4.3| Fuzzy Programming Approach for Bi-Level LPPs

In the decision making context, each DM is interested in maximizing his or her own objective function, the optimal solution of each DM when calculated in isolation would be considered as the best solution and the associated objective value can be considered as the aspiration level of the corresponding fuzzy goal because both the DMs are interested of maximizing their own objective functions over the same feasible region defined by the system of constraints. Let x_i^B be the best (optimal) solution of the i^{th} level DM. It is quite natural that objective values which are equal to or larger than $f_i^B = f_i(x_i^B) = max\ f_i(x)$, $i = 1,2.$, $x \in S$ should be absolutely satisfactory to the i^{th} level DM. If the individual best (optimal) solution $x_i^B, i = 1,2$ are the same, then a satisfactory optimal solution of the system is reached. However, this rarely happens due to the conflicting nature of the objectives. To obtain a satisfactory solution, higher level DM should give some tolerance (relaxation) and the relaxation of decision of the higher level DM depends on the needs, desires and practical situations in the decision making situation .Then the fuzzy goals take the form $f_i(x) \lesssim f_i\left(x_i^B\right)$, $i = 1,2, x_i \cong x_i^B$.

To build membership functions, goals and tolerance should be determined first. However, they could hardly be determined without meaningful supporting data. Using the individual best solutions, we find the values of all the objective functions at each best solution and construct a payoff matrix

$$\begin{bmatrix} & f_1(x) & f_2(x) \\ x_1^0 & f_1(x_1^0) & f_2(x_1^0) \\ x_2^0 & f_1(x_2^0) & f_2(x_2^0) \end{bmatrix}.$$

The maximum value of each column $\left(f_i\left(x_i^0\right)\right)$ gives upper tolerance limit or aspired level of achievement for the ith objective function where $f_i^u = f_i\left(x_i^0\right) = \max f_i\left(x_i^0\right)$, $i = 1,2$.

The minimum value of each column gives lower tolerance limit or lowest acceptable level of achievement for the i[th] objective function where $f_i^L = min\ f_i(x_i^0)$, $i = 1.2$. For the maximization type objective function, the upper tolerance limit f_t^u, $t = 1,2$, are kept constant at their respective optimal values calculated in isolation but the lower tolerance limit f_i^L are changed. The idea being that $f_i(x) \rightarrow f_t^u$, then the fuzzy objective goals take the form $f_i(x) \lesssim f_i(x_i^u)$, $i =$ 1,2. And the fuzzy goal for the control vector x_i is obtained a $x_i \cong x_i^u$. Now, in the decision situation, it is assumed that all DMs that are up to i^{th} motivation to cooperate each other to make a balance of decision powers, and they agree to give a possible relaxation of their individual optimal decision. The i^{th} level DM must adjust his/her goal by assuming the lowest acceptable level of achievement f_i^L based on indefiniteness of the decentralized organization. Thus, all values of $f_i(x) \geq f_t^u$ are absolutely acceptable (desired) to objective function $f_i(x)$ satisfactory to the ith level DM. All values of $f_i(x)$f with $f_i(x) \leq f_t^L$ are absolutely unacceptable (undesired) to the objective function $f_i(x)$ for $i = 1,2$. Based on this interval of tolerance, we can establish the following linear membership functions for the defined fuzzy goals as *Fig.1* below.

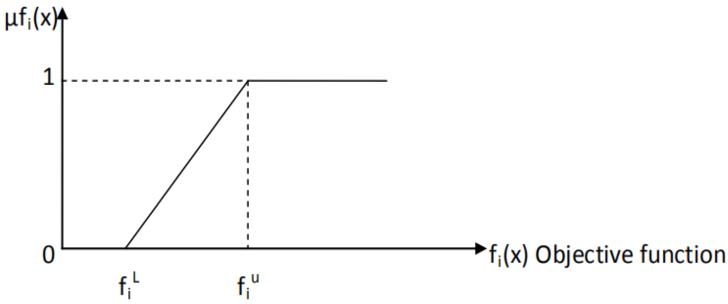

Fig. 1. Membership function of maximization-type objective function.

$$\mu_i(f_i(x)) = \begin{cases} 1, & \text{if } f_i(x) \geq f_i^u, \\ \dfrac{f_i(x) - f_i^L}{f_i^u \geq f_i^L}, & \text{if } f_i^L \leq f_i(x) \leq f_i^u, i = 1,2 \\ 0, & \text{if } f_i(x) \leq f_i^L. \end{cases} \tag{10}$$

By identifying the membership functions $\mu_1(f_1(x))$and $\mu_2(f_2(x))$for the objective functions $f_1(x)$ and $f_2(x)$, and following the principle of the fuzzy decision by Bellman and Zadeh, the original bi-level linear programming *Problem (9)* can be interpreted as the membership function maxmin problem defined by:

$$\begin{aligned} &\max \min\{\mu_i(f_i(x)),\ \ i = 1,2\}, \\ &\text{s.t.} \\ &A_1 x_1 + A_2 x_2 \leq b,\ \ x_1, x_2 \geq 0. \end{aligned} \tag{11}$$

Then the linear membership functions for decision vector x_1 can be formulated as:

$$\mu_{x1}(f_1(x)) = \begin{cases} \dfrac{x_1 - (x_1^0 - e_1^-)}{e_1^-}, & \text{if } x_1^0 - e_1^- \leq x_1 \leq x_1^0 \\ \dfrac{(x_1^0 + e_1^+) - x_1}{e_1^+}, & \text{if } x_1^0 \leq x_1 \leq (x_1^0 + e_1^+) \\ 0, & \text{if otherwise.} \end{cases} \tag{12}$$

Where x_1^0 is the optimal solution of first level DM;

e_1^- the negative tolerance value on x_1;

e_1^+ the positive tolerance value on x_1.

To derive an overall satisfactory solution to the membership function maximization *Problem (11)*, we first find the maximizing decision of the fuzzy decision proposed by Bellman and Zadeh [22]. Namely, the following problem is solved for obtaining a solution which maximizes the smaller degree of satisfaction between those of the two decision makers:

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$$\max\ \min\{\mu_1(f_1(x)),\ \mu_2(f_2(x)),\ \mu_{x1}(x_1)\},$$
$$\text{s.t.} \tag{13}$$
$$A_1x_1 + A_2x_2 \le b,\quad x_1, x_2 \ge 0.$$

By introducing an auxiliary variable λ, this problem can be transformed into the following equivalent problem:

$$\max\ \lambda,$$
$$\text{s.t.}\ \mu_1(f_1(x)) \ge \lambda,$$
$$\mu_2(f_2(x)) \ge \lambda, \tag{14}$$
$$\mu_{x1}(x_1) \ge \lambda,$$
$$A_1x_1 + A_2x_2 \le b,\quad x_1, x_2 \ge 0.$$

Solving *Problem (14)*, we can obtain a solution which maximizes the smaller satisfactory degree between those of both decision makers. It should be noted that if the membership function$\mu_i(f_i(x)), i = 1.2$ are linear membership functions such as *Eq. (10)*, *Problem (14)* becomes a linear programming problem. Let x^*denotes an optimal solution to *Problem (14)*. Then we define the satisfactory degree of both decision makers under the constraints as

$$\lambda^* = \min\{\mu_1(f_1(x^*)),\ \mu_2(f_2(x^*))\}. \tag{15}$$

If DM1 is satisfied with the optimal solution x^*, it follows that the optimal solution x^*becomes a satisfactory solution; however DM1 is not always satisfied with the solution x^*. It is quite natural to assume that DM1 specifies (decides) the minimal satisfactory level $\delta \in [0,1]$ for his membership function subjectively. Consequently, DM2 optimizes his objective under the new constraints as the following problem:

$$\max\ \mu_2(f_2(x)),$$
$$\text{s.t.} \tag{16}$$
$$\mu_1(f_1(x)) \le \delta$$
$$A_1x_1 + A_2x_2 \le b,\quad x_1, x_2 \ge 0.$$

If an optimal solution to *Problem (16)* exists, it follows that DM1 obtains a satisfactory solution having a satisfactory degree larger than or equal to the minimal satisfactory level specified (decided) by DM1's own self. However, the larger the minimal satisfactory level is assessed, the smaller DM2's satisfactory degree becomes. Consequently, a relative difference between the satisfactory degrees of DM1 and DM2 becomes larger than it is feared that overall satisfactory balance between both levels cannot be maintained. To take account of overall satisfactory balance between both levels, DM1 needs to compromise (agree) with DM2 on DM1' s own minimal satisfactory level. To do so, the following ratio of the satisfactory degree of DM2 to that of DM1 is defined as:

$$\Delta = \frac{\mu_2\left(f_2(x^*)\right)}{\mu_1\left(f_1(x^*)\right)}. \tag{17}$$

This is originally introduced by Lai [6].

Let $\Delta > \Delta^L$ denote the lower bound and the upper bound of Δ specified by DM1. If $\Delta > \Delta^U, i.e\ \mu_2\left(f_2(x^*)\right) > \Delta^U \mu_1\left(f_1(x^*)\right)$, then DM1 updates (improves) the minimal satisfactory level δ by increasing δ . Then DM1 obtains a larger satisfactory degree and DM2 accepts a smaller satisfactory degree. Conversely, if $\Delta > \Delta^L, i.e\ \mu_2\left(f_2(x^*)\right) < \Delta^i \mu_1\left(f_1(x^*)\right)$, then DM1 updates the minimal satisfactory level δ by decreasing δ, and DM1 accepts a smaller satisfactory degree and DM2 obtains a larger satisfactory degree.

At an iteration l , let $\mu_1\left(f_1(x^l)\right)$, $\mu_2\left(f_2(x^l)\right), \lambda^l$ and $\Delta^l = \frac{\mu_2\left(f_2(x_1^l)\right)}{\mu_1\left(f_1(x^l)\right)}$ denote DM1's and DM2's satisfactory degrees, a satisfactory degree of both levels and the ratio of satisfactory degrees between both DMs, respectively, and let a corresponding solution be l^x at the iteration. The iterated interactive process terminates if the following two conditions are satisfied and DM1 concludes the solution as a satisfactory solution.

4.3.1| Termination conditions of the interactive processes for bi-level linear programming problems

DM1's satisfactory degree is larger than or equal to the minimal satisfactory level δ specified by DM1, i.e. $\mu_1\left(f_1(x^l)\right) \geq \delta$.

The ratio $\Delta^l$of satisfactory degrees lies in the closed interval between the lower and upper bounds specified by DM1, i.e. $\Delta^l \in [\Delta\ min, \Delta max]$.

Condition (i) is DM1's required condition for solutions, and Condition (ii) is provided in order to keep overall satisfactory balance between both levels. Unless the conditions are satisfied simultaneously, DM1 needs to update the minimal satisfactory level δ.

Procedure for updating the minimal satisfactory level δ.

If Condition (i) is not satisfied, then DM1 decreases the minimal satisfactory level by δ.

If the ratio Δ^l exceeds its upper bound, then DM1 increases the minimal satisfactory level δ. Conversely, if the ratio $\Delta^l$is below its lower bound, then DM1 decreases the minimal satisfactory level δ.

4.4| Algorithm of Interactive Fuzzy Programming for BLPPs

Step 1. Find the solution of the first level and second level independently with the same feasible set given.

Step 2. Do these solutions coincide?

- *If yes, an optimal solution is reached.*
- *If No, go to Step 3.*

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Step 3. Define a fuzzy goal, construct a payoff matrix, and then find upper tolerance limit f_t^u and lower tolerance limit f_t^L.

Step 4. Build member ship functions for maximization objective functions $\mu f_i(f_i(x))$ and decision vector x_1 using *Eqs. (8)* and *(10)*, respectively.

Step 5. set $\ell = 1$ and solve the auxiliary *Problems (14)*. If DM1 is satisfied with the optimal solution, the solution becomes a satisfactory solution x^*. Otherwise, ask DM1 to specify (decide) the minimal satisfactory level δ together with the lower and the upper bounds $[\Delta\ min, \Delta\ max]$ of the ratio of satisfactory degrees Δ^l with the satisfactory degree $\lambda^*$of both decision makers and the related information about the solution in mind.

Step 6. Solve *Problem (16)*, in which the satisfactory degree of DM1 is maximized under the condition that the satisfactory degree of DM1 is larger than or equal to the minimal satisfactory level δ, and then an optimal solution x^l to *Problem (16)* is proposed to DM1 together with λ^l, $\mu_1(f_1(x^l))$, $\mu_2(f_2(x^l))$ and Δ^l.

Step 7. If the solution x^l satisfies the termination conditions and DM1 accepts it, then the procedure stops, and the solution x^l is determined to be a satisfactory solution.

Step 8. Ask DM1 to revise the minimal satisfactory level δ in accordance with the procedure for updating minimal satisfactory level. Return to Step 7.

Example 2. Solve (Linear BLPP)

$$
\begin{aligned}
&\max f_1(x) = 5x_1 + 6x_2 + 4x_3 + 2x_4,\\
&x_1, x_2.\\
&\text{Where } x_3, x_4 \text{ solves}\\
&\max f_2(x) = 8x_1 + 9x_2 + 2x_3 + 4x_4,\\
&x_3, x_4,\\
&\text{s. t.}\\
&3x_1 + 2x_2 + x_3 + 3x_4 \le 40,\\
&x_1 + 2x_2 + x_3 + 2x_4 \le 30,\\
&2x_1 + 4x_2 + x_3 + 2x_4 \le 35,\\
&x_1, x_2, x_3, x_4 \ge 0.
\end{aligned}
\tag{18}
$$

Solution.

Step 1. Find the solution of the top-level and lower-level independently with the same feasible set. i.e.

$$\max f_1(x) = 5x_1 + 6x_2 + 4x_3 + 2x_4,$$
$$\begin{aligned} &\text{s.t.}\\ &3x_1 + 2x_2 + x_3 + 3x_4 \le 40,\\ &x_1 + 2x_2 + x_3 + 2x_4 \le 30,\\ &2x_1 + 4x_2 + x_3 + 2x_4 \le 35,\\ &x_1, x_2, x_3, x_4 \ge 0. \end{aligned} \tag{19}$$

Then we find the optimal solution

$f_1 = 125\ at\, x_1^0 = (5, 0, 25, 0)$;

$f_2 = 118.125\ at\, x_2^0 = (11.25, 3.125, 0, 0)$;

But this is not a satisfactory solution (since $x_1^0 \neq x_2^0$).

Step 2. Define fuzzy goals, construct the payoff matrix and we need to find the upper and lower tolerance limit.

Objective function as: $f_1 \lesssim 125, f_2 \lesssim 118.125$.

Decision variables as: $x_1 \cong 5, x_2 \cong 0$;

$$\text{Payoff matrix}=\begin{bmatrix} & f_1(x_1^0) & f_2(x_2^0) \\ x_1^0 & 125 & 90 \\ x_2^0 & 75 & 118.125 \end{bmatrix}.$$

Upper tolerance limits are $f_1^u = 125, f_2^u \lesssim 118.125$.

Lower tolerance limits are $f_1^L = 75, f_2^L \lesssim 90$.

Step 3. Build membership functions for:

Objective functions as

$$\mu f_1\big(f_1(x)\big) = \begin{cases} 1, & \text{if } f_1(x) \ge 125 \\ \dfrac{f_1(x) - 75}{125 - 75}, & \text{if } 75 \le f_1(x) \le 125 \\ 0, & \text{if } f_1(x) \le 75 \end{cases}.$$

Decision variable function as

$$\mu f_2\big(f_2(x)\big) = \begin{cases} 1, & \text{if } f_2(x) \ge 118.125 \\ \dfrac{f_2(x) - 90}{118.125 - 90}, & \text{if } 90 \le f_2(x) \le 119.125. \\ 0, & \text{if } f_2(x) \le 90 \end{cases}$$

Let the upper level DM specifies (decides) $x_1 = 5$ with 2.5 (negative) and 2.5 (positive) tolerances and $x_2 = 0$ with 0 (negative) and 3 (positive) tolerance values.

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$$\mu x_1(x_1) = \begin{cases} \dfrac{x_1 - (5 - 2.5)}{2.5}, & \text{if } 2.5 \le x_1 \le 5 \\ \dfrac{(5 + 2.5) - x_1}{2.5}, & \text{if } 5 \le x_1 \le 7.5' \\ 0, & \text{otherwise} \end{cases}$$

$$\mu x_2(x_2) = \begin{cases} x_2, & \text{if } x_2 \le 3 \\ \dfrac{3 - x_2}{3}, & \text{if } 0 \le x_2 \le 3 \\ 0, & \text{otherwise} \end{cases}.$$

Step 4. Solve the auxiliary problem

$$\begin{aligned} & \max \lambda, \\ & \text{s.t.} \\ & \mu f_1\big(f_1(x)\big) \ge \lambda, \\ & \mu f_2\big(f_2(x)\big) \ge \lambda, \\ & \mu x_1(x_1) \ge \lambda, \\ & 3x_1 + 2x_2 + x_3 + 3x_4 \le 40, \\ & x_1 + 2x_2 + x_3 + 2x_4 \le 30, \\ & 2x_1 + 4x_2 + x_3 + 2x_4 \le 35, \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned} \tag{20}$$

The result of the first iteration including an optimal solution to the problem is

$$x_1^1 = 6.41, x_2^1 = 1.95, x_3^1 = 10.52, x_4^1 = 1.42, \text{ and } \lambda^1 = 0.316, f_1^1(x) = 88.67, f_2^1(x) = 95.55, \mu_1\big(f_1(x)\big) = 0.2734.$$

Suppose that DM1 is not satisfied with the solution obtained in iteration 1, and then let him specify (decide) the minimal satisfactory level at $\delta = 0.3$ and the bounds of the ratio at the interval $[\Delta min, \Delta\, max] = [0.3, 0.4]$, taking account of the result of the first iteration. Then, the problem with the minimal satisfactory level is written as:

$$\begin{aligned} & \max \mu f_2(f_2(x)), \\ & \text{s.t.} \\ & \mu f_1(f_1(x)) \ge 0.3, \\ & x \in S. \end{aligned} \tag{21}$$

Applying simplex algorithm, the result of the second iteration including an optimal solution to *Problem (21)* is

$$x_1^2 = 6.71, x_2^2 = 2.05, x_3^2 = 10.52, x_4^2 = 1.42,$$

and

$$\lambda^2 = 0.316,$$

$$f_1^2(x) = 90.77, f_2^2(x) = 98.85, \mu_{f1}\big(f_1(x)\big) = 0.3154, \tag{22}$$

and

$$\Delta^2 = 0.3165.$$

Therefore, this solution satisfies the termination conditions.

5| Conclusion

The fuzzy mathematically programming approach is simple to implement, interactive and applicable to BLPP. The satisfactory solution obtained is realistic. We can take any membership function other than linear. The results will hold good, however, the problem will become a non linear programming problem. We observe that even though the decision making process is from higher to lower level, the lower level becomes the most important. This is because the decision vector under the control of the lower level DM is not given any tolerance limits. Hence this decision vector either remains unchanged or closest to its valued obtained in isolation. But at higher level, the decision vectors are given some tolerance and hence they are free to move within the tolerance limits. The tolerance levels can also be considered as variables and if the DMs cooperate then the entire system as a whole can be optimized. We can easily apply the same approach to non linear BLPPs.

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An Overview of Data Envelopment Analysis Models in Fuzzy Stochastic Environments

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Abstract

One of the appropriate and efficient tools in the field of productivity measurement and evaluation is data envelopment analysis, which is used as a non-parametric method to calculate the efficiency of decision-making units. Today, the use of data envelopment analysis technique is expanding rapidly and is used in the evaluation of various organizations and industries such as banks, postal service, hospitals, training centers, power plants, refineries, etc. In real-world problems, the values observed from input and output data are often ambiguous and random. To solve this problem, data envelopment analysis in stochastic fuzzy environment was proposed. Although the DEA has many advantages, one of the disadvantages of this method is that the classic DEA does not actually give us a definitive conclusion and does not allow random changes in input and output. In this paper, we review some of the proposed models in data envelopment analysis with fuzzy and random inputs and outputs.

Keywords: Decision-Making, Efficiency, Stochastic fuzzy DEA.

1 | Introduction

Data envelopment analysis is a linear programming method whose basic purpose is to compare and evaluate the performance of a number of identical decision-making units that have different amounts of inputs used and outputs produced. Data Envelopment Analysis (DEA) models used in evaluating the performance of the unit under study can use two separate approaches: reducing the amount of inputs without decreasing the amount of outputs, increasing the outputs without increasing the amount of inputs.

In real world problems, inputs and outputs are considered vague and random. In fact, decision makers may face a specific hybrid environment where there is fuzziness and randomness in the problem.

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Hatami-Marbini et al. classified the fuzzy DEA methods in the literature into five general groups [1], the tolerance approach [2] and [3], the α-level based approach, the fuzzy ranking approach [4] and [5], the possibility approach [6], and the fuzzy arithmetic approach [7]. Among these approaches, the α-level based approach is probably the most popular fuzzy DEA model in the literature. This approach generally tries to transform the FDEA model into a pair of parametric programs for each α-level. Kao and Liu, one of the most cited studies in the α-level approach's category, used Chen and Klein [8] method for ranking fuzzy numbers to convert the FDEA model to a pair of parametric mathematical programs for the given level of α [9]. Saati et al. proposed a fuzzy CCR model as a possibilistic programming problem and changed it into an interval programming problem by means of the α-level based approach [10]. Parameshwaran et al. proposed an integrated fuzzy analytic hierarchy process and DEA approach for the service performance measurement [11]. Puri and Yadav [12] applied the suggested methodology by Saati et al. [10] to solve fuzzy DEA model with undesirable outputs. Khanjani et al. [13] proposed fuzzy free disposal hull models under possibility and credibility measures. Momeni et al. used fuzzy DEA models to address the impreciseness and ambiguity associated with input and output data in supply chain performance evaluation problems [14]. Payan evaluated the performance of DMUs with fuzzy data by using the common set of weights based on a linear program [15]. Aghayi et al. formulated a model to measure the efficiency of DMUs with interval inputs and outputs based on common sets weights [16].

In recent years, several scholars work on DEA with fuzzy set extension. For example, Edalatpanah et al. [17] for the first time established triangular single-valued neutrosophic data envelopment analysis with application to hospital performance. He also presented data envelopment analysis based on triangular neutrosophic numbers [18]; see also [19]-[22].

In this research, some models of data envelopment analysis with fuzzy and random data will be mentioned.

2| Existing Models

In this section, we review the proposed models in a random fuzzy environment with undesirable outputs.

Nasseri et al. [23] proposed a DEA-based method for evaluating the efficiencies of DMUs that not only depicts the impact of undesirable output on the performance of units, but also evaluated the efficiencies of DMUs with stochastic inputs and fuzzy stochastic outputs.

They considered n DMUs, indexed by $j=1,\dots,n$. Each of with consumes m fuzzy random inputs, denoted by $\tilde{\tilde{x}}_{ij}=(\tilde{x}_{ij},x^{\alpha}_{ij},x^{\beta}_{ij})_{LR}$, i=1,…,m to produce $s=s_1+s_2$ fuzzy random outputs, denoted by $\tilde{\tilde{y}}^{g}_{rj}=(\tilde{y}^{g}_{rj},y^{g,\alpha}_{rj},y^{g,\beta}_{rj})$, $r==1,\dots,s_1$ as desirable outputs and $\tilde{\tilde{y}}^{b}_{pj}=(\tilde{y}^{b}_{pj},y^{b,\alpha}_{pj},y^{b,\beta}_{pj})_{LR}$, p=1,…,$s_2$ as undesirable outputs. Let the random parameters $\tilde{x}_{ij}$, $\tilde{y}^{g}_{rj}$ and $\tilde{y}^{b}_{pj}$, denoted by $N(x_{ij},\sigma_{ij})$, $N(y^{g}_{rj},\sigma^{g}_{rj})$, $N(y^{b}_{pj},\sigma^{b}_{pj})$, respectively, be normally distributed. Here, $x_{ij}(y^{g}_{rj},y^{b}_{pj})$ and $\sigma_{ij}(\sigma^{g}_{rj},\sigma^{b}_{pj})$, are the mean value and the variance for $\tilde{x}_{ij}(\tilde{y}^{g}_{rj},\tilde{y}^{b}_{pj})$, respectively.

The Chance-Constrained Programming (CCP) developed by Cooper et al. [24] is a stochastic optimization approach suitable for solving optimization problems with uncertain parameters. Using the concepts of CCP and probability (possibility) of stochastic (fuzzy) events, the deterministic model will be as follows:

$$
\begin{aligned}
& E_k^{Pos}(\gamma,\delta) = \max \ \varphi \\
& \text{s.t.} \\
& \varphi - \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rk}^g + R^{-1}(\delta) y_{rk}^{g,\beta}) + \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pk}^b - L^{-1}(\delta) y_{pk}^{b,\alpha}) \le \sigma_k^y \boldsymbol{f}_{1-\gamma}^{-1}, \\
& \sum_{i=1}^{m} v_i(\tilde{x}_{ik} + R^{-1}(\delta) x_{ik}^{\beta}) + \sigma_k^x \boldsymbol{f}_{1-\gamma}^{-1} \ge 1, \\
& \sum_{i=1}^{m} v_i(\tilde{x}_{ik} - L^{-1}(\delta) x_{ik}^{\alpha}) - \sigma_k^x \boldsymbol{f}_{1-\gamma}^{-1} \le 1, \\
& \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g + R^{-1}(\delta_j) y_{rj}^{g,\beta}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b - L^{-1}(\delta_j) y_{pj}^{b,\alpha}) - \sigma_j^y \boldsymbol{f}_{1-\gamma}^{-1} \ge 0, \quad \forall j \\
& \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g - L^{-1}(\delta) y_{rj}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b + R^{-1}(\delta) y_{pj}^{b,\beta}) \\
& - \sum_{i=1}^{m} v_i(\tilde{x}_{ij} + R^{-1}(\delta) x_{ij}^{\beta}) - \sigma_j^A \boldsymbol{f}_{1-\gamma}^{-1} \le 0, \quad \forall j \\
& u_r^g \ge 0 \forall r, \ u_p^b \ge 0 \forall p, \ v_i \ge 0 \forall i.
\end{aligned} \tag{1}
$$

This model is always feasible as the traditional DEA-UO model.

Then, they presented the CCR-UO model with fuzzy probability-necessity constraints. They considered n DMUs with m fuzzy stochastic inputs, s_1 desirable and s_2 undesirable outputs. The deterministic model will be as following:

$$
\begin{aligned}
& E_k^{Nec}(\gamma,\delta) = \max \ \varphi \\
& \text{s.t.} \\
& \varphi \le 1, \\
& \varphi \le \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rk}^g - L^{-1}(1-\delta) y_{rk}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pk}^b + R^{-1}(1-\delta) y_{pk}^{b,\beta}) + \sigma_k^y \boldsymbol{f}_{1-\gamma}^{-1}, \\
& \sum_{i=1}^{m} v_i(\tilde{x}_{ik} - L^{-1}(1-\delta) x_{ik}^{\alpha}) + \sigma_k^x \boldsymbol{f}_{1-\gamma}^{-1} \ge 1, \\
& \sum_{i=1}^{m} v_i(\tilde{x}_{ik} + R^{-1}(1-\delta) x_{ik}^{\beta}) - \sigma_k^x \boldsymbol{f}_{1-\gamma}^{-1} \le 1, \\
& \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g - L^{-1}(1-\delta) y_{rj}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b + R^{-1}(1-\delta) y_{pj}^{b,\beta}) + \sigma_j^y \boldsymbol{f}_{1-\gamma}^{-1} \ge 0, \quad \forall j \\
& \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g + R^{-1}(1-\delta) y_{rj}^{g,\beta}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b - L^{-1}(1-\delta) y_{pj}^{b,\alpha}) - \sum_{i=1}^{m} v_i(\tilde{x}_{ij} - L^{-1}(1-\delta) x_{ij}^{\alpha}) - \sigma_j^A \boldsymbol{f}_{1-\gamma}^{-1} \le 0, \quad \forall j \\
& \sigma_k^x = (\sum_{i=1}^{m} v_i^2 \sigma_{ik}^2)^{1/2} \\
& \sigma_j^y = (\sum_{r=1}^{s_1} (u_r^g)^2 (\sigma_{rj}^g)^2 + \sum_{p=1}^{s_2} (u_p^b)^2 (\sigma_{pj}^b)^2)^{1/2}, \quad \forall j \\
& \sigma_j^A = (\sum_{r=1}^{s_1} (u_r^g)^2 (\sigma_{rj}^g)^2 + \sum_{p=1}^{s_2} (u_p^b)^2 (\sigma_{pj}^b)^2 + \sum_{i=1}^{m} v_i^2 \sigma_{ik}^2)^{1/2}, \quad \forall j \\
& u_r^g \ge 0 \forall r, \ u_p^b \ge 0 \forall p, \ v_i \ge 0 \forall i.
\end{aligned} \tag{2}
$$

Analogously to the previous models, the corresponding fuzzy probability-credibility CCR-UO model was introduced. Thus, this model for $\delta \le 0.5$ and $\delta \ge 0.5$ can be transformed into the following two models:

For $\delta \le 0.5$:

$$
\begin{aligned}
& E_k^{Cr}(\gamma,\delta) = \max \ \varphi \\
& \text{s.t.} \\
& \varphi \le 1, \\
& \varphi \le \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rk}^g + R^{-1}(2\delta) y_{rk}^{g,\beta}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pk}^b - L^{-1}(2\delta) y_{pk}^{b,\alpha}) + \sigma_k^y \boldsymbol{f}_{1-\gamma}^{-1}, \\
& \sum_{i=1}^{m} v_i(\tilde{x}_{ik} + R^{-1}(2\delta) x_{ik}^{\beta}) + \sigma_k^x \boldsymbol{f}_{1-\gamma}^{-1} \ge 1, \\
& \sum_{i=1}^{m} v_i(\tilde{x}_{ik} - L^{-1}(2\delta) x_{ik}^{\alpha}) - \sigma_k^x \boldsymbol{f}_{1-\gamma}^{-1} \le 1,
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
&\sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g + R^{-1}(2\delta)y_{rj}^{g,\beta}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b - L^{-1}(2\delta)y_{pj}^{b,\alpha}) + \sigma_j^y \boldsymbol{f}_{1-\gamma}^{-1} \geq 0, \quad \forall j\\
&\sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g - L^{-1}(2\delta)y_{rj}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b + R^{-1}(2\delta)y_{pj}^{b,\beta}) - \sum_{i=1}^{m} v_i(\tilde{x}_{ij} + R^{-1}(2\delta)x_{ij}^{\beta}) - \sigma_j^A \boldsymbol{f}_{1-\gamma}^{-1} \leq 0, \quad \forall j\\
&\sigma_k^x = (\sum_{i=1}^{m} v_i^2 \sigma_{ik}^2)^{1/2}\\
&\sigma_j^y = (\sum_{r=1}^{s_1} (u_r^g)^2(\sigma_{rk}^g)^2 + \sum_{p=1}^{s_2} (u_p^b)^2(\sigma_{pk}^g)^2)^{1/2}, \quad \forall j\\
&\sigma_j^A = (\sum_{r=1}^{s_1} (u_r^g)^2(\sigma_{rj}^g)^2 + \sum_{p=1}^{s_2} (u_p^b)^2(\sigma_{pj}^b)^2 + \sum_{i=1}^{m} v_i^2 \sigma_{ij0}^2)^{1/2}, \quad \forall j\\
&u_r^g \geq 0 \forall r, \ u_p^b \geq 0 \forall p, \ v_i \geq 0 \forall i.
\end{aligned}
$$

And for $\delta \leq 0/5$:

$$
\begin{aligned}
&E_k^{Cr}(\gamma,\delta) = \max \ \varphi\\
&\text{s.t.}\\
&\varphi \leq 1,\\
&\varphi \leq \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rk}^g - L^{-1}(2(1-\delta))y_{rk}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pk}^b + R^{-1}(2(1-\delta))y_{pk}^{b,\beta}) + \sigma_k^y \boldsymbol{f}_{1-\gamma}^{-1},\\
&\sum_{j=1}^{n} v_i(\tilde{x}_{ik} - L^{-1}(2(1-\delta))x_{ik}^{\alpha}) + \sigma_k^x \boldsymbol{f}_{1-\gamma}^{-1} \geq 1,\\
&\sum_{j=1}^{n} v_i(\tilde{x}_{ik} + R^{-1}(2(1-\delta))x_{ik}^{\beta}) - \sigma_k^x \boldsymbol{f}_{1-\gamma}^{-1} \leq 1,\\
&\sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g - L^{-1}(2(1-\delta))y_{rj}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b + R^{-1}(2(1-\delta))y_{pj}^{b,\beta}) + \sigma_j^y \boldsymbol{f}_{1-\gamma}^{-1} \geq 0, \quad \forall j\\
&\sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g + R^{-1}(2(1-\delta))y_{rj}^{g,\beta}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b - L^{-1}(2(1-\delta))y_{pj}^{b,\alpha}) - \sum_{i=1}^{m} v_i(\tilde{x}_{ij} - L^{-1}(2(1-\delta))x_{ij}^{\alpha}) - \sigma_j^A \boldsymbol{f}_{1-\gamma}^{-1} \leq 0, \quad \forall j\\
&\sigma_k^x = (\sum_{i=1}^{m} v_i^2 \sigma_{ik}^2)^{1/2}\\
&\sigma_j^y = (\sum_{r=1}^{s_1} (u_r^g)^2(\sigma_{rk}^g)^2 + \sum_{p=1}^{s_2} (u_p^b)^2(\sigma_{pk}^g)^2)^{1/2}, \quad \forall j\\
&\sigma_j^A = (\sum_{r=1}^{s_1} (u_r^g)^2(\sigma_{rj}^g)^2 + \sum_{p=1}^{s_2} (u_p^b)^2(\sigma_{pj}^b)^2 + \sum_{i=1}^{m} v_i^2 \sigma_{ij}^2)^{1/2}, \quad \forall j\\
&u_r^g \geq 0 \forall r, \ u_p^b \geq 0 \forall p, \ v_i \geq 0 \forall i.
\end{aligned} \tag{4}
$$

In 2016, Nasseri et al. [24] proposed a new model of fuzzy stochastic DEA with input-oriented primal data. In this model, the properties and characteristics of the extended normal distribution are used. They considered n DMUs, each unit consumes m fuzzy stochastic inputs, denoted by $\tilde{\tilde{x}}_{ij} = (x_{ij}^m, x_{ij}^{\alpha}, x_{ij}^{\beta})_{LR}$, $i=1,\ldots,m$, $j=1,\ldots,n$, and produces s fuzzy stochastic outputs, denoted by $\tilde{\tilde{y}}_{rj} = (y_{rj}^m, y_{rj}^{\alpha}, y_{rj}^{\beta})_{LR}$, $r=1,\ldots,s$, $j=1,\ldots,n$. Also, they considered x_{ij}^m and y_{rj}^m, denoted by $x_{ij}^m \sim N(x_{ij}, \sigma_{ij}^2)$ and $y_{rj}^m \sim N(y_{rj}, \sigma_{rj}^2)$ be normally distributed. Therefore, $x_{ij}(y_{rj})$ and $\sigma_{ij}^2(\sigma_{rj}^2)$ are the mean and the variance of $x_{ij}^m(y_{rj}^m)$ for DMU_j, respectively. Each unit has an extended normal distribution as $\tilde{\tilde{x}}_{ij} \sim \overline{N}(\overline{x}_{ij}, \sigma_{ij})$ with $\overline{x}_{ij} = (x_{ij}, x_{ij}^{\alpha}, x_{ij}^{\beta})$ and $\tilde{\tilde{y}}_{rj} \sim \overline{N}(\overline{y}_{rj}, \sigma_{rj})$ with $\overline{y}_{rj} = (y_{rj}, y_{rj}^{\alpha}, y_{rj}^{\beta})$. Finally, the final model is as follows:

$$
\begin{aligned}
&E_K^T(\delta,\gamma) = \max \ \varphi\\
&\text{s.t.}\\
&\varphi \leq \sum_{r=1}^{s} \hat{y}_{rk},\\
&\sum_{i=1}^{m} \hat{x}_{ik} = 1,\\
&\sum_{r=1}^{s} \hat{y}_{rj} - \sum_{i=1}^{m} \hat{x}_{ij} \leq 0 \quad \forall j,
\end{aligned} \tag{5}
$$

$$u_r(y_{rj} - L^{-1}(\delta)y_{rj}^{\alpha} - \sigma_{rj}\phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{y}_{rj} \le u_r(y_{rj} + R^{-1}(\delta)y_{rj}^{\beta} + \sigma_{rj}\phi^{-1}_{1-\frac{\gamma}{2}}), \forall r, j$$
$$v_i(x_{rj} - L^{-1}(\delta)x_{rj}^{\alpha} - \sigma_{ij}\phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{x}_{ij} \le v_i(x_{ij} + R^{-1}(\delta)x_{ij}^{\beta} + \sigma_{ij}\phi^{-1}_{1-\frac{\gamma}{2}}), \forall i, j$$
$$u_r, v_i \ge 0.$$

Theorem 1. Assume that ξ is a fuzzy random vector, and g_j are real-valued continuous functions for i=1,2,…,n. We have:

The possibility $pos\{g_j(\xi(\omega)) \le 0,\ j = 1,...,n\}$ is a random variable.

The necessity $Nec\{g_j(\xi(\omega)) \le 0,\ j = 1,...,n\}$ is a random variable.

The credibility $Cr\{g_j(\xi(\omega)) \le 0,\ j = 1,...,n\}$ is a random variable.

Lemma 1. Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be two fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in \left[0,1\right]$, $Pos\{\bar{\lambda}_1 \ge \bar{\lambda}_2\} \ge \alpha$ if and only if $\lambda^R_{1,\alpha} \ge \lambda^R_{2,\alpha}$ and $Nec\{\bar{\lambda}_1 \ge \bar{\lambda}_2\} \ge \alpha$ if and only if $\lambda^L_{1,1-\alpha} \ge \lambda^R_{2,\alpha}$. Where $\lambda^L_{1,\alpha}, \lambda^R_{1,\alpha}$ and $\lambda^L_{2,\alpha}, \lambda^R_{2,\alpha}$ are the left and the right side extreme points of the α-level sets $\bar{\lambda}_1$ and $\bar{\lambda}_2$, respectively, and $Pos\{\bar{\lambda}_1 \ge \bar{\lambda}_2\}$ and $Nec\{\bar{\lambda}_1 \ge \bar{\lambda}_2\}$ present the degree of possibility and necessity, respectively.

Definition 1. A DMU is said to be probabilistic-possibility, probabilistic-necessity and probabilistic-credibility (γ, δ)-efficient if the objective function of *Models (1)- (4)*, φ, is equal to unity at the threshold level (γ, δ); otherwise, it is said to be probabilistic-possibility, probabilistic-necessity and probabilistic-credibility (γ, δ)-inefficient.

Theorem 2. Consider $E_k^{T,Pos}(\delta, \gamma)$ as the objective function value of DMU_k, then

$E_k^{T,Pos}(\delta_1, \gamma) \ge E_k^{T,Pos}(\delta_2, \gamma)$ and $E_k^{T,Pos}(\delta, \gamma_1) \ge E_k^{T,Pos}(\delta, \gamma_2)$ where $\delta_1 \le \delta_2$ and $\gamma_1 \le \gamma_2$.

The model related to $E_k^{T,Pos}(\delta, \gamma)$ is feasible for any δ and γ.

3| Conclusion

A DEA model basically draws three critical elements: the model specification, the reference set itself, and the definition of the production possibility set. Starting from the latter, the production possibility set can either be defined as complete and known (like in conventional DEA) or as potentially extending beyond or excluding the reference set (like in stochastic DEA). The reference set, the very observations that form the engine of the non-parametric approach, can be either precise (as in conventional DEA), outcomes of stochastic processes (as in stochastic frontier analysis), or imprecise (as in the fuzzy DEA models).

Classic DEA models were originally formulated for optimal inputs and outputs, although undesirable outputs may also appear during production, which should be minimized. In addition, in the real world, there are dimensions and uncertainties in the data. Although DEA has many advantages, one of the

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disadvantages of this method is that in fact the classic DEA does not lead us to a definite conclusion and does not allow random changes in input and output.

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A Study on Fundamentals of Refined Intuitionistic Fuzzy Set with Some Properties

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Abstract

Zadeh conceptualized the theory of fuzzy set to provide a tool for the basis of the theory of possibility. Atanassov extended this theory with the introduction of intuitionistic fuzzy set. Smarandache introduced the concept of refined intuitionistic fuzzy set by further subdivision of membership and non-membership value. The meagerness regarding the allocation of a single membership and non-membership value to any object under consideration is addressed with this novel refinement. In this study, this novel idea is utilized to characterize the essential elements e.g. subset, equal set, null set, and complement set, for refined intuitionistic fuzzy set. Moreover, their basic set theoretic operations like union, intersection, extended intersection, restricted union, restricted intersection, and restricted difference, are conceptualized. Furthermore, some basic laws are also discussed with the help of an illustrative example in each case for vivid understanding.

Keywords: Fuzzy set, Intuitionistic fuzzy set, Refined intuitionistic fuzzy set.

1 | Introduction

Zadeh [1] introduced the concept of fuzzy set for the first time in 1965 which covers all weak aspects of the classical set theory. In fuzzy set, the membership value is allocated from the interval [0, 1] to all the elements of the universe under consideration. Zadeh [2] used his own concept as a basis for a theory of possibility. Dubois and Prade [3] and [4] established relationship between fuzzy sets and probability theories and also derive monotonicity property for algebraic operations performed between random set-valued variables. Ranking fuzzy numbers in the setting of possibility theory was done by Dubois and Prade [5]. This concept was used by Liang et al. in data analysis, similarities measures in fuzzy sets were discussed by Beg and Ashraf [6]-[8].

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Set difference and symmetric difference of fuzzy sets were established by Vemuri et al., after that, Neog and Sut [9] extended the work to complement of an extended fuzzy set. A lot of work is done by researchers in fuzzy mathematics and its hybrids [10]-[16].

In some real life situations, the values are in the form of intervals due to which it is hard to allocate a membership value to the element of the universe of discourse. Therefore, the concept of interval-valued is introduced which proved a very powerful tool in this area.

In 1986, Atanassov [17] and [18] introduced the concept of intuitionistic fuzzy set in which the membership value and non-membership value is allocated from the interval $[0,1]$ to all the elements of the universe under consideration. It is the generalization of the fuzzy set. The invention of intuitionistic fuzzy set proved very important tool for researchers. Ejegwa et al. [19] discussed about operations, algebra, model operators and normalization on intuitionistic fuzzy sets. Szmidt and Kacprzyk [20] gave geometrical representation of an intuitionistic fuzzy set is a point of departure for our proposal of distances between intuitionistic fuzzy sets and also discussed properties. Szmidt and Kacprzyk [21] also discussed about non-probabilistic-type entropy measure for these sets and used it for geometrical interpretation of intuitionistic fuzzy sets. Proposed measure in terms of the ratio of intuitionistic fuzzy cardinalities was also defined and discussed. Ersoy and Davvaz [22] discussed the basic definitions and properties of intuitionistic fuzzy Γ- hyperideals of a Γ- semi-hyperring with examples are introduced and described some characterizations of Artinian and Noetherian Γ- semi hyper ring. Bustince and Burillo [23] proved that vague sets are intuitionistic fuzzy sets. A lot of work is done by researchers in intuitionistic fuzzy environment and its hybrids [24]-[33].

In 2019, Smarandache defined the concept of refined intuitionistic fuzzy set [34]. In this paper, we extend the concept to refined intuitionistic fuzzy set and defined some fundamental concepts and aggregation operations of refined intuitionistic fuzzy set.

Imprecision is a critical viewpoint in any decision making procedure. Various tools have been invented to deal with the uncertain environment. Perhaps the important tool in managing with imprecision is intuitionistic fuzzy sets. Besides, the most significant thing is that in real life scenario, it is not sufficient to allocate a single membership and non-membership value to any object under consideration. This inadequacy is addressed with the introduction of refined intuitionistic fuzzy set. Having motivation from this novel concept, essential elements, set theoretic operations and basic laws are characterized for refined intuitionistic fuzzy set in this work.

The remaining article is outlined in such a way that the Section 2 recalls some basic definitions along with illustrative example. Section 3 explains basic notions of Refined Intuitionistic Fuzzy Set (RIFS) including subset, equal set, null set and complement set along with their examples for the clear understanding. Section 4 explains the aggregation operations of RIFS with the help of example, Section 5 gives some basic laws of RIFS and in the last, Section 6 concludes the work and gives the future directions.

2| Preliminaries

In this section, some basic concepts of Fuzzy Set (FS), Intuitionistic Fuzzy Set (IFS) and RIFS are discussed.

Let us consider $\breve{U}$ be a universal set, N be a set of natural numbers, $\check{I}$ represent the interval $[0,1]$, T_{η}^{ω} denotes the degree of sub-truth of type $\omega = 1,2,3, \dots, \alpha$ and F_{η}^{λ} denotes the degree of sub-falsity of type $\lambda = 1,2,3, \dots, \beta$ such that α and β are natural numbers. An illustrative example is considered to understand these entire basic concepts throughout the paper.

Definition 1. [1, 2] The fuzzy set $\breve{\eta}_f = \{<\breve{\delta}, \alpha_{\breve{\eta}_f}(\breve{\delta})> | \breve{\delta} \in \breve{U}\}$ on $\breve{U}$ such that $\alpha_{\breve{\eta}_f}(\breve{\delta}): \breve{U} \to \breve{I}$ where $\alpha_{\breve{\eta}_f}(\breve{\delta})$ describes the membership of $\breve{\delta} \in \breve{U}$.

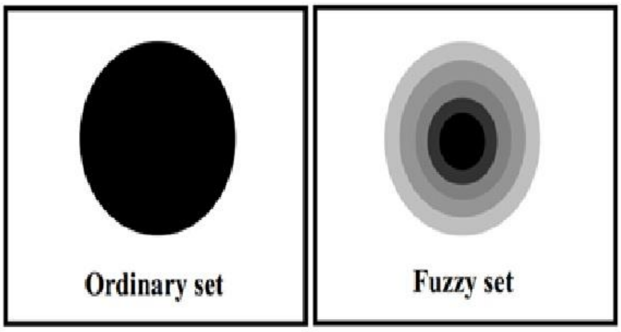

Fig. 1. Representation of fuzzy set.

Example 1. Hamna wants to purchase a dress for farewell party event of her university. She expected to purchase such dress which meets her desired requirements according to the event. Let $\breve{U} = \{\breve{B}_1, \breve{B}_2, \breve{B}_3, \breve{B}_4\}$, be different well-known brands of clothes in Pakistan such that

$\breve{B}_1$ = Ideas Gul Ahmad;
$\breve{B}_2$ = Khaadi;
$\breve{B}_3$ = Nishat Linen;
$\breve{B}_4$ = Junaid jamshaid.

Then fuzzy set $\breve{\eta}_f$ on the universe $\breve{U}$ is written in such a way that $\breve{\eta}_f = \left\{\begin{matrix} <\breve{B}_1, 0.45>, <\breve{B}_2, 0.57>, \\ <\breve{B}_3, 0.6>, <\breve{B}_4, 0.64> \end{matrix}\right\}$.

Definition 2. [18]. An IFS $\breve{\eta}_{IFS}$ on $\breve{U}$ is given by $\breve{\eta}_{IFS} = \{<\breve{\delta}, T_{\breve{\eta}}(\breve{\delta}), F_{\breve{\eta}}(\breve{\delta})> | \breve{\delta} \in \breve{U}\}$,

where $T_{\breve{\eta}}(\breve{\delta}), F_{\breve{\eta}}(\breve{\delta}): \breve{U} \to P([0,1])$, respectively, with the condition $sup\, T_{\breve{\eta}}(\breve{\delta}) + sup\, F_{\breve{\eta}}(\breve{\delta}) \leq 1$.

Example 2. Consider the illustrative example, and then the intuitionistic fuzzy set $\breve{\eta}_{IFS}$ on the universe $\breve{U}$ is given as $\breve{\eta}_{IFS} = \{<\breve{B}_1, 0.75, 0.14>, <\breve{B}_2, 0.57, 0.2>, <\breve{B}_3, 0.6, 0.3>, <\breve{B}_4, 0.64, 0.16>\}$.

Definition 3. [34] A RIFS $\breve{\eta}_{RIFS}$ on $\breve{U}$ is given by $\breve{\eta}_{RIFS} = \{<\breve{\delta}, T^{\omega}_{\breve{\eta}}(\breve{\delta}), F^{\lambda}_{\breve{\eta}}(\breve{\delta})>: \omega \in N^{\alpha}_1, \lambda \in N^{\beta}_1, \alpha + \beta \geq 3, \breve{\delta} \in \breve{U}\}$, where $\alpha, \beta \in \breve{I}$ such that $T^{\omega}_{\breve{\eta}}, F^{\lambda}_{\breve{\eta}} \subseteq \breve{I}$, respectively, with the condition

$$\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}}(\breve{\delta}) + \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}}(\breve{\delta}) \leq 1.$$

It is denoted by $(\breve{\delta}, \breve{G})$, where $\breve{G} = (T^{\omega}_{\breve{\eta}}, F^{\lambda}_{\breve{\eta}})$.

Example 3. Consider the illustrative example, then the RIFS $\breve{\eta}_{RIFS}$ can be written in such a way that

$$\breve{\eta}_{RIFS} = \{<\breve{B}_1, (0.5, 0.4), (0.3, 0.25)>, <\breve{B}_2, (0.35, 0.3), (0.15, 0.1)>,$$
$$<\breve{B}_3, (0.35, 0.25), (0.3, 0.2)>, <\breve{B}_4, (0.6, 0.1), (0.12, 0.2)>\}.$$

3| Basic Notions of RIFS

In this section, some basic notions of subset, equal sets, null set and complement set for RIFS are introduced.

Definition 4. Refined intuitionistic fuzzy subset

Let $\breve{\eta}1_{RIFS} = (\breve{\delta},\ \breve{G}_1)$ and $\breve{\eta}2_{RIFS} = (\breve{\delta},\ \breve{G}_2)$ be two RIFS, then $\breve{\eta}1_{RIFS} \subseteq\ \breve{\eta}2_{RIFS}$, if

$$\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}1}(\breve{\delta}) \leq \sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}2}(\breve{\delta}),\quad \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}1}(\breve{\delta}) \geq \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}2}(\breve{\delta})\ \ \forall\ \ \breve{\delta}\ \in \breve{U}.$$

Remark 1. If

$$\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}1}(\breve{\delta}) < \sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}2}(\breve{\delta}),\quad \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}1}(\breve{\delta}) > \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}2}(\breve{\delta})\ \ \forall\ \ \breve{\delta}\ \in \breve{U}.$$

Then it is denoted by $(\breve{\delta}, \breve{G}_1) \subset (\breve{\delta}, \breve{G}_2)$.

Suppose $(\breve{\delta}, \breve{G}^i_1) \subset (\breve{\delta}, \breve{G}^i_2)$ be two families of RIFS, then $(\breve{\delta}, \breve{G}^i_1)$ is called family of refined intuitionistic fuzzy subset of $(\breve{\delta}, \breve{G}^i_2)$, if $\breve{G}^i_1 \subset \breve{G}^i_2$ and

$$\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}1}(\breve{\delta}) < \sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}2}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}1}(\breve{\delta}) > \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}2}(\breve{\delta}), \forall\ \breve{\delta} \in \breve{U}.$$

We denote it by $(\breve{\delta}, \breve{G}^i_1) \subset (\breve{\delta}, \breve{G}^i_2) \forall \quad i = 1, 2, 3, \dots, n.$

Example 4. Consider the illustrative example, let $\breve{\eta}1_{RIFS}$ and $\breve{\eta}2_{RIFS}$ be two RIFS such that

$$\breve{\eta}1_{RIFS} = \{ < \breve{B}_1, (0.35, 0.1), (0.22, 0.19) >, < \breve{B}_2, (0.25, 0.03), (0.15, 0.19) >,$$
$$< \breve{B}_3, (0.2, 0.1), (0.2, 0.24) >, < \breve{B}_4, (0.3, 0.4), (0.06, 0.04) > \},$$

and

$$\breve{\eta}2_{RIFS} = \{ < \breve{B}_1, (0.38, 0.11), (0.2, 0.14) >, < \breve{B}_2, (0.45, 0.04), (0.1, 0.14) >,$$
$$< \breve{B}_3, (0.3, 0.2), (0.01, 0.06) >, < \breve{B}_4, (0.31, 0.41), (0.01, 0.011) > \}.$$

Then from above equations, it is clear that $\breve{\eta}1_{RIFS} \subseteq\ \breve{\eta}2_{RIFS}$.

Definition 5. Equal refined intuitionistic fuzzy sets

Let $\breve{\eta}1_{RIFS} = (\breve{\delta},\ \breve{G}_1)$ and $\breve{\eta}2_{RIFS} = (\breve{\delta},\ \breve{G}_2)$ be two RIFS, then $\breve{\eta}1_{RIFS} = \breve{\eta}2_{RIFS}$, if $\breve{\eta}1_{RIFS} \subseteq \breve{\eta}2_{RIFS}$ and $\breve{\eta}2_{RIFS} \subseteq\ \breve{\eta}1_{RIFS}$.

Example 5. Consider the illustrative example, let $\breve{\eta}1_{RIFS}$ and $\breve{\eta}2_{RIFS}$ be two RIFS such that

$$\breve{\eta}1_{RIFS} = (\breve{\delta}, \breve{G}_1) = \{ < \breve{B}_1, (0.4, 0.5), (0.03, 0.04) >, < \breve{B}_2, (0.5, 0.4), (0.05, 0.04) >,$$
$$< \breve{B}_3, (0.5, 0.2), (0.01, 0.06) >, < \breve{B}_4, (0.3, 0.4), (0.06, 0.04) > \},$$

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and

$\breve{\eta}2_{RIFS} = (\breve{\delta}, \breve{G}_2) = \{ < \breve{B}_1, (0.4, 0.5), (0.03, 0.04) >, < \breve{B}_2, (0.5,0.4), (0.05,0.04) >,$
$< \breve{B}_3, (0.5,0.2), (0.01,0.06) >, < \breve{B}_4, (0.3,0.4), (0.06,0.04) > \}.$

Then from above equations, it is clear that $\breve{\eta}1_{RIFS} = \breve{\eta}2_{RIFS}$.

Definition 6. Null refined intuitionistic fuzzy set

Let RIFS $(\breve{\delta}, \breve{G})$ is said to be null RIFS if

$$\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}}^{\omega}(\breve{\delta}) = 0, \quad \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}}^{\lambda}(\breve{\delta}) = 0, \qquad \forall \quad \breve{\delta} \in \breve{U}.$$

It is denoted by $(\breve{\delta}, \breve{G})_{null}$.

Example 6. Consider the illustrative example, the null RIFS is given as

$(\breve{\delta}, \breve{G}) = \{ < \breve{B}_1, (0, 0), (0, 0) >, < \breve{B}_2, (0,0), (0,0) >,$

$< \breve{B}_3, (0,0), (0,0) >, < \breve{B}_4, (0,0), (0,0) > \}.$

Definition 7. Complement of refined intuitionistic fuzzy set

The complement of RIFS$(\breve{\delta}, \breve{G})$ is denoted by $(\breve{\delta}, \breve{G}^c)$ and is defined that if

$$\sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}^c}^{\omega}(\breve{\delta}) = \sum_{\lambda=1}^{\beta} \sup F_{\breve{\eta}}^{\lambda}(\breve{\delta}), \qquad \sum_{\lambda=1}^{\beta} \sup F_{\breve{\eta}^c}^{\lambda}(\breve{\delta}) = \sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}}^{\omega}(\breve{\delta}), \qquad \forall \quad \breve{\delta} \in \breve{U}.$$

Remark 2. The complement of family of RIFS$(\breve{\delta}, \breve{G}^c)$ is denoted by $(\breve{\delta}, \breve{G}^c)$ and is defined in a way that if

$$\sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}_i^c}^{\omega}(\breve{\delta}) = \sum_{\lambda=1}^{\beta} \sup F_{\breve{\eta}}^{\lambda}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F_{\breve{\eta}_i^c}^{\lambda}(\breve{\delta}) = \sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}}^{\omega}(\breve{\delta}), \forall\ \ i = 1,2,3, \dots, n.$$

Example 7. Consider the illustrative example, if there is a RIFS $\breve{\eta}_{RIFS}$ given as

$\breve{\eta}_{RIFS} = \{ < \breve{B}_1, (0.2, 0.1), (0.3, 0.35) >, < \breve{B}_2, (0.05,0.34), (0.45,0.04) >,$
$< \breve{B}_3, (0.01,0.6), (0.1,0.02) >, < \breve{B}_4, (0.3,0.04), (0.12,0.2) > \}.$

Then the complement of RIFS$\breve{\eta}_{RIFS}$ given as

$\breve{\eta}_{RIFS} = \{ < \breve{B}_1(0.3, 0.35), (0.2, 0.1) >, < \breve{B}_2, (0.45,0.04), (0.05,0.34) >,$
$< \breve{B}_3, (0.1,0.02), (0.01,0.6) >, < \breve{B}_4, (0.12,0.2), (0.3,0.04) > \}.$

4 | Aggregation Operators of RIFS

In this section, union, intersection, extended intersection, restricted union, restricted intersection and restricted difference of RIFS is defined with the help of illustrative example.

Definition 8. Union of two RIFS

The union of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cup (\breve{\delta},\ \breve{G}_2)$ and it isdefined as $\left(\breve{\delta},\ \breve{G}_1\right) \cup \left(\breve{\delta},\ \breve{G}_2\right) = (\breve{\delta},\ \breve{\Upsilon})$, where $\breve{\Upsilon} = \breve{G}_1 \cup \breve{G}_2$, and truth and falsemembership of $(\breve{\delta},\ \breve{\Upsilon})$ is defined in such a way that

$$T_{\breve{\Upsilon}}\left(\breve{\delta}\right) = \max\left(\sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}1}^{\omega}\left(\breve{\delta}\right), \sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}2}^{\omega}\left(\breve{\delta}\right)\right),$$

$$F_{\breve{\Upsilon}}\left(\breve{\delta}\right) = min\left(\sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}1}^{\lambda}\left(\breve{\delta}\right), \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}2}^{\lambda}\left(\breve{\delta}\right)\right).$$

Remark 3. The union of two families of RIFS $(\breve{\delta},\ \breve{G}_1^i)$ and $(\breve{\delta},\ \breve{G}_2^i)$is denoted by $(\breve{\delta},\ \breve{G}_1^i) \cup (\breve{\delta},\ \breve{G}_2^i)$ and it is defined as $\left(\breve{\delta},\ \breve{G}_1^i\right) \cup \left(\breve{\delta},\ \breve{G}_2^i\right) = (\breve{\delta},\ \breve{\Upsilon}^i)$, where $\breve{\Upsilon}^i = \breve{G}_1^i \cup \breve{G}_2^i$, $i = 1,2,3, \dots, n$, and truth and false membership of $(\breve{\delta},\ \breve{\Upsilon}^i)$ is defined in such a way that

$$T_{\breve{\Upsilon}^i}\left(\breve{\delta}\right) = max\left(\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}1}^{\omega}\left(\breve{\delta}\right), \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}2}^{\omega}\left(\breve{\delta}\right)\right),$$

$$F_{\breve{\Upsilon}^i}\left(\breve{\delta}\right) = min\left(\sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}1}^{\lambda}\left(\breve{\delta}\right), \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}2}^{\lambda}\left(\breve{\delta}\right)\right).$$

Example 8. Consider the illustrative example, suppose that

$$(\breve{\delta},\ \breve{G}_1) = \{ <\breve{B}_1, (0.13,0.19),(0.24, 0.1)>, <\breve{B}_2, (0.2,0.25), (0.15,0.24)>,$$

$$<\breve{B}_3, (0.1,0.36), (0.34,0.12)>, <\breve{B}_4, (0.16,0.14), (0.23,0.37)> \},$$

and

$$(\breve{\delta},\ \breve{G}_2) = \{ <\breve{B}_1, (0.2,0.3),(0.3, 0.15)>,$$

$$<\breve{B}_2, (0.32,0.38), (0.1,0.04)>,$$
$$<\breve{B}_3, (0.01,0.16), (0.5,0.2)>, <\breve{B}_4, (0.26,0.15), (0.12,0.2)> \},$$

be two RIFS. Then the union of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$$(\breve{\delta},\ \breve{\Upsilon}), = \{ <\breve{B}_1, (0.2, 0.3),(0.24,0.1)>, <\breve{B}_2, (0.32,0.38), (0.1,0.04)>,$$
$$<\breve{B}_3, (0.1,0.36), (0.34,0.12)>, <\breve{B}_4, (0.26,0.15), (0.12,0.2)> \}.$$

Definition 9. Intersection of two RIFS

The intersection of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cap (\breve{\delta},\ \breve{G}_2)$ and it is defined as $\left(\breve{\delta},\ \breve{G}_1\right) \cap \left(\breve{\delta},\ \breve{G}_2\right) = (\breve{\delta},\ \breve{\Upsilon})$, where $\breve{\Upsilon} = \breve{G}_1 \cap \breve{G}_2$, and truth and falsemembership of $(\breve{\delta},\ \breve{\Upsilon})$ is defined in such a way that

$$T_{\breve{\Upsilon}}(\breve{\delta}) = \min\left(\sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}1}^{\omega}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}2}^{\omega}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}}(\breve{\delta}) = \max\left(\sum_{\lambda=1}^{\beta} \sup F_{\breve{\eta}1}^{\lambda}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F_{\breve{\eta}2}^{\lambda}(\breve{\delta})\right).$$

Remark 4. The intersection of two families of RIFS $(\breve{\delta},\ \breve{G}_1^i)$ and $(\breve{\delta},\ \breve{G}_2^i)$is denoted by $(\breve{\delta},\ \breve{G}_1^i) \cap (\breve{\delta},\ \breve{G}_2^i)$ and it is defined as$\left(\breve{\delta},\ \breve{G}_1^i\right) \cap \left(\breve{\delta},\ \breve{G}_2^i\right) = (\breve{\delta},\ \breve{\Upsilon}^i)$, where $\breve{\Upsilon}^i = \breve{G}_1^i \cap \breve{G}_2^i$, $i = 1,2,3, \ldots, n$, and truth and false membership of $(\breve{\delta},\ \breve{\Upsilon}^i)$ is defined in such a way that

$$T_{\breve{\Upsilon}^i}(\breve{\delta}) = \min\left(\sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}1}^{\omega}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}2}^{\omega}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}^i}(\breve{\delta}) = \max\left(\sum_{\lambda=1}^{\beta} \sup F_{\breve{\eta}1}^{\lambda}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F_{\breve{\eta}2}^{\lambda}(\breve{\delta})\right).$$

Example 9. Consider the illustrative example, suppose that

$$(\breve{\delta},\ \breve{G}_1) = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$
$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \},$$

and

$$(\breve{\delta},\ \breve{G}_2) = \{ < \breve{B}_1, (0.2,0.3),(0.3, 0.15)>,$$
$$< \breve{B}_2, (0.32,0.38), (0.1,0.04) >,$$
$$< \breve{B}_3, (0.01,0.16), (0.5,0.2) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \},$$

be two RIFS. Then the intersection of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$$(\breve{\delta},\ \breve{\Upsilon}) = \{ < \breve{B}_1, (0.13, 0.19),(0.24, 0.1)>,$$
$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.01,0.16), (0.5,0.2) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}.$$

Definition 10. Extended intersection of two RIFS

The intersection of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cap_{\varepsilon} (\breve{\delta},\ \breve{G}_2)$ and it is defined as $\left(\breve{\delta},\ \breve{G}_1\right) \cap_{\varepsilon} \left(\breve{\delta},\ \breve{G}_2\right) = (\breve{\delta},\ \breve{\Upsilon})$, where $\breve{\Upsilon} = \breve{G}_1 \cup \breve{G}_2$, and truth and falsemembership of $(\breve{\delta},\ \breve{\Upsilon})$ is defined in such a way that

$$T_{\breve{\Upsilon}}(\breve{\delta}) = \min\left(\sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}1}^{\omega}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup\ T_{\breve{\eta}2}^{\omega}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}}(\breve{\delta}) = \max\left(\sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}2}(\breve{\delta})\right).$$

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Remark 5. The extended intersection of two families of RIFS $(\breve{\delta}, \breve{G}^{i}_{1})$ and $(\breve{\delta}, \breve{G}^{i}_{2})$is denoted by $(\breve{\delta}, \breve{G}^{i}_{1}) \cap_{\varepsilon} (\breve{\delta}, \breve{G}^{i}_{2})$ and it isdefined as$\left(\breve{\delta}, \breve{G}^{i}_{1}\right) \cap_{\varepsilon} \left(\breve{\delta}, \breve{G}^{i}_{2}\right) = (\breve{\delta}, \breve{\Upsilon}^{i})$, where $\breve{\Upsilon}^{i} = \breve{G}^{i}_{1} \cup \breve{G}^{i}_{2}$, $i = 1,2,3, \dots, n$, and truth and false membership of $(\breve{\delta}, \breve{\Upsilon}^{i})$ is defined in such a way that

$$T_{\breve{\Upsilon}^{i}}(\breve{\delta}) = \min\left(\sum_{\omega=1}^{\alpha} \sup T^{\omega}_{\breve{\eta}1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup T^{\omega}_{\breve{\eta}2}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}^{i}}(\breve{\delta}) = \max\left(\sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}2}(\breve{\delta})\right).$$

Example 10. Consider the illustrative example, suppose that

$(\breve{\delta}, \breve{G}_{1}) = \{ < \breve{B}_{1}, (0.13, 0.19),(0.24,0.1)>,$

$< \breve{B}_{2}, (0.2,0.25), (0.15,0.24) >,$
$< \breve{B}_{3}, (0.1,0.36), (0.34,0.12) > \},$

and

$(\breve{\delta}, \breve{G}_{2}) = \{< \breve{B}_{3}, (0.01,0.16), (0.5,0.2) >, < \breve{B}_{4}, (0.26,0.15), (0.12,0.2) > \}$, be two RIFS. Then the extended intersection of RIFS $(\breve{\delta}, \breve{G}_{1})$ and $\left(\breve{\delta}, \breve{G}_{2}\right)$ is given as

$(\breve{\delta}, \breve{\Upsilon}) = \{ < \breve{B}_{1}, (0.13,0.19),(0.24,0.1)>, < \breve{B}_{2}, (0.2,0.25), (0.15,0.24) >,$
$< \breve{B}_{3}, (0.01,0.16), (0.5,0.2) >, < \breve{B}_{4}, (0.26,0.15), (0.12,0.2) > \}.$

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Definition 11. Restricted union of two RIFS

The restricted union of two RIFS $(\breve{\delta}, \breve{G}_{1})$ and $(\breve{\delta}, \breve{G}_{2})$is denoted by $\left(\breve{\delta}, \breve{G}_{1}\right) \cup_{R} (\breve{\delta}, \breve{G}_{2})$ and it isdefined as $\left(\breve{\delta}, \breve{G}_{1}\right) \cup_{R} \left(\breve{\delta}, \breve{G}_{2}\right) = (\breve{\delta}, \breve{\Upsilon})$, where $\breve{\Upsilon} = \breve{G}_{1} \cap_{R} \breve{G}_{2}$, and truth and falsemembership of $(\breve{\delta}, \breve{\Upsilon})$ is defined in such a way that

$$T_{\breve{\Upsilon}}(\breve{\delta}) = \max\left(\sum_{\omega=1}^{\alpha} \sup T^{\omega}_{\breve{\eta}1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup T^{\omega}_{\breve{\eta}2}(\breve{\delta})\right).$$

$$F_{\breve{\Upsilon}}(\breve{\delta}) = \min\left(\sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}2}(\breve{\delta})\right).$$

Remark 6. The restricted union of two families of RIFS $(\breve{\delta}, \breve{G}^{i}_{1})$ and $(\breve{\delta}, \breve{G}^{i}_{2})$is denoted by

$(\breve{\delta}, \breve{G}^{i}_{1}) \cup_{R} (\breve{\delta}, \breve{G}^{i}_{2})$ and it is defined as $\left(\breve{\delta}, \breve{G}^{i}_{1}\right) \cup_{R} \left(\breve{\delta}, \breve{G}^{i}_{2}\right) = (\breve{\delta}, \breve{\Upsilon}^{i})$, where $\breve{\Upsilon}^{i} = \breve{G}^{i}_{1} \cap_{R} \breve{G}^{i}_{2}$, $i = 1,2,3, \dots, n$, and truth and false membership of $(\breve{\delta}, \breve{\Upsilon}^{i})$ is defined in such a way that

$$T_{\breve{\Upsilon}^i}(\breve{\delta}) = \max\left(\sum_{\omega=1}^{\alpha} \sup\ T^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup\ T^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}^i}(\breve{\delta}) = \min\left(\sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Example 11. Consider the illustrative example, suppose that

$$(\breve{\delta},\ \breve{G}_1) = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) > \},$$

and $(\breve{\delta}, \breve{G}_2) = \{< \breve{B}_3, (0.01,0.16), (0.5,0.2) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \}$, be two RIFS. Then the restricted union of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$$(\breve{\delta}, \breve{\Upsilon}) = \{< \breve{B}_3, (0.1,0.36), (0.34,0.12) > \}.$$

Definition 12. Restricted intersection of two RIFS

The restricted intersection of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cap_R (\breve{\delta},\ \breve{G}_2)$ and it isdefined as $(\breve{\delta},\ \breve{G}_1) \cap_R (\breve{\delta},\ \breve{G}_2) = (\breve{\delta},\ \breve{\Upsilon})$, where $\breve{\Upsilon} = \breve{G}_1 \cap_R \breve{G}_2$, and truth and false membership of $(\breve{\delta},\ \breve{\Upsilon})$ is defined in such a way that

$$T_{\breve{\Upsilon}}(\breve{\delta}) = \min\left(\sum_{\omega=1}^{\alpha} \sup\ T^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup\ T^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}}(\breve{\delta}) = \max\left(\sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Remark 7. The restricted intersection of two families of RIFS $(\breve{\delta},\ \breve{G}_1^i)$ and $(\breve{\delta},\ \breve{G}_2^i)$is denoted by

$(\breve{\delta},\ \breve{G}_1^i) \cap_R (\breve{\delta},\ \breve{G}_2^i)$ and it isdefined as $(\breve{\delta},\ \breve{G}_1^i) \cap_R (\breve{\delta},\ \breve{G}_2^i) = (\breve{\delta},\ \breve{\Upsilon}^i)$, where $\breve{\Upsilon}^i = \breve{G}_1^i \cap_R \breve{G}_2^i$, $i = 1,2,3, \ldots, n$, and truth and false membership of $(\breve{\delta},\ \breve{\Upsilon}^i)$ is defined in such a way that

$$T_{\breve{\Upsilon}^i}(\breve{\delta}) = \min\left(\sum_{\omega=1}^{\alpha} \sup\ T^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup\ T^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}^i}(\breve{\delta}) = \max\left(\sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} \sup F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Example 12. Consider the illustrative example, suppose that

$$(\breve{\delta},\ \breve{G}_1) = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) > \},$$

and $(\breve{\delta}, \breve{G}_2) = \{< \breve{B}_2, (0.32,0.38), (0.1,0.04) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \}$,

be two RIFS. Then the restricted intersection of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$$(\breve{\delta}, \breve{\Upsilon}) = \{< \breve{B}_2, (0.2,0.25), (0.15,0.24) > \}.$$

Definition 13. Restricted difference of two RIFS

The restricted difference of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $\left(\breve{\delta},\ \breve{G}_1\right) -_R (\breve{\delta},\ \breve{G}_2)$ and it is defined as $\left(\breve{\delta},\ \breve{G}_1\right) -_R \left(\breve{\delta},\ \breve{G}_2\right) = (\breve{\delta},\ \breve{\Upsilon})$, where $\breve{\Upsilon} = \breve{G}_1 -_R \breve{G}_2$.

Example 13. Consider the illustrative example, suppose that

$$(\breve{\delta},\ \breve{G}_1) = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \},$$

and

$$(\breve{\delta},\ \breve{G}_2) = \{ < \breve{B}_1, (0.2,0.3),(0.3, 0.15)>, < \breve{B}_2, (0.32,0.38), (0.1,0.04) >,$$

$$< \breve{B}_3, (0.01,0.16), (0.5,0.2) > \},$$

be two RIFS. Then the restricted difference of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$$(\breve{\delta}, \breve{\Upsilon}) = \{< \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}.$$

5 | Some Basic Laws of RIFS

In this section, we prove some basic fundamental laws including idempotent law, identity law, domination law, De-Morgan law and commutative law with the help of illustrative example.

5.1 | Idempotent Law

$$\left(\breve{\delta},\breve{G}\right) \cup \left(\breve{\delta},\breve{G}\right) = \left(\breve{\delta},\breve{G}\right) = \left(\breve{\delta},\breve{G}\right) \cup_R \left(\breve{\delta},\breve{G}\right).$$

$$\left(\breve{\delta},\breve{G}\right) \cap \left(\breve{\delta},\breve{G}\right) = \left(\breve{\delta},\breve{G}\right) = \left(\breve{\delta},\breve{G}\right) \cap_\varepsilon \left(\breve{\delta},\breve{G}\right).$$

Example 14. To prove (1) law, we consider illustrative example. For this, suppose that

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$$(\breve{\delta},\ \breve{G}) = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}.$$

One can observe

$$\left(\breve{\delta},\ \breve{G}\right) \cup \left(\breve{\delta},\breve{G}\right) = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \} = \left(\breve{\delta},\breve{G}\right) = \left(\breve{\delta},\breve{G}\right) \cup_R \left(\breve{\delta},\breve{G}\right).$$

Similarly, we can prove (2).

5.2 | Identity Law

$$\left(\breve{\delta},\breve{G}\right) \cup \breve{\varnothing} = \left(\breve{\delta},\breve{G}\right) = \left(\breve{\delta},\breve{G}\right) \cup_R \breve{\varnothing}.$$

$$\left(\breve{\delta},\breve{G}\right) \cap \breve{U} = \left(\breve{\delta},\breve{G}\right) = \left(\breve{\delta},\breve{G}\right) \cap_\varepsilon \breve{U}.$$

Example 15. To prove (1) law, we consider illustrative example. For this, suppose that

$$(\breve{\delta},\ \breve{G}) = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}.$$

One can observe

$$\left(\breve{\delta},\ \breve{G}\right) \cup \breve{\varnothing} = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \} = \left(\breve{\delta},\breve{G}\right) = \left(\breve{\delta},\breve{G}\right) \cup_R \breve{\varnothing}.$$

Similarly, we can *Prove (2).*

5.3 | Domination Law

$$\left(\breve{\delta},\breve{G}\right) \cup \breve{U} = \breve{U} = \left(\breve{\delta},\breve{G}\right) \cup_R \breve{U}.$$

$$\left(\breve{\delta},\breve{G}\right) \cap \breve{\varnothing} = \breve{\varnothing} = \left(\breve{\delta},\breve{G}\right) \cap_\varepsilon \breve{\varnothing}.$$

Example 16. To *Prove (1)* law, we consider illustrative example. For this, suppose that

$$(\breve{\delta},\ \breve{G}) = \{ < \breve{B}_1, (0.13, 0.19), (0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}.$$

One can observe

$$\left(\breve{\delta},\ \breve{G}\right) \cup \breve{U} = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \} \cup \breve{U}$$

$$= \breve{U} = \left(\breve{\delta},\breve{G}\right) \cup_R \breve{U}.$$

Similarly, we can *Prove (2).*

5.4 | De-Morgan Law

$$\left(\left(\breve{\delta},\breve{G}_1\right) \cup \left(\breve{\delta},\breve{G}_2\right)\right)^c = \left(\breve{\delta},\breve{G}_1\right)^c \cap_\varepsilon \left(\breve{\delta},\breve{G}_2\right)^c.$$

$$\left(\left(\breve{\delta},\breve{G}_1\right) \cap_\varepsilon \left(\breve{\delta},\breve{G}_2\right)\right)^c = \left(\breve{\delta},\breve{G}_1\right)^c \cup \left(\breve{\delta},\breve{G}_2\right)^c.$$

Example 17. To prove (1) law, we consider illustrative example. For this, suppose that L.H.S is

$$\left(\breve{\delta},\breve{G}_1\right) \cup \left(\breve{\delta},\breve{G}_2\right) = \{ < \breve{B}_1, (0.2, 0.3),(0.24,0.1)>,$$

$$< \breve{B}_2, (0.32,0.38), (0.1,0.04) >,$$
$$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \}.$$

Then

$$\left(\left(\breve{\delta},\breve{G}_1\right) \cup \left(\breve{\delta},\breve{G}_2\right)\right)^c = \{ < \breve{B}_1, (0.24,0.1), (0.2,0.3)>,$$

$$< \breve{B}_2, (0.1,0.04), (0.32,0.38) >,$$
$$< \breve{B}_3, (0.34,0.12), (0.1,0.36) >, < \breve{B}_4, (0.12,0.2), (0.26,0.15) >\}.$$

Now consider R.H.S.

$$\left(\breve{\delta},\breve{G}_1\right)^c \cap_\varepsilon \left(\breve{\delta},\breve{G}_2\right)^c = \{ < \breve{B}_1, (0.24,0.1), (0.2,0.3)>,$$

$$< \breve{B}_2, (0.1,0.04), (0.32,0.38) >,$$
$$< \breve{B}_3, (0.34,0.12), (0.1,0.36) >, < \breve{B}_4, (0.12,0.2), (0.26,0.15) >\}.$$

From this, it is clear that L.H.S.=R.H.S. Similarly, we can prove (2).

5.5 | Commutative Law

$$\left(\breve{\delta},\breve{G}_1\right) \cup \left(\breve{\delta},\breve{G}_2\right) = \left(\breve{\delta},\breve{G}_2\right) \cup \left(\breve{\delta},\breve{G}_1\right).$$

$$\left(\breve{\delta},\breve{G}_1\right) \cup_R \left(\breve{\delta},\breve{G}_2\right) = \left(\breve{\delta},\breve{G}_2\right) \cup_R \left(\breve{\delta},\breve{G}_1\right).$$

$$\left(\breve{\delta},\breve{G}_1\right) \cap \left(\breve{\delta},\breve{G}_2\right) = \left(\breve{\delta},\breve{G}_2\right) \cap \left(\breve{\delta},\breve{G}_1\right).$$

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$$\left(\breve{\delta},\breve{G}_1\right)\cap_\varepsilon\left(\breve{\delta},\breve{G}_2\right)=\left(\breve{\delta},\breve{G}_2\right)\cap_\varepsilon\left(\breve{\delta},\breve{G}_1\right).$$

Example 18. To *Prove (1)* law, we consider illustrative example. For this, suppose that

L.H.S:

$$\left(\breve{\delta},\breve{G}_1\right)\cup\left(\breve{\delta},\breve{G}_2\right)=\{<\breve{B}_1,(0.2,0.3),(0.24,0.1)>,$$

$$<\breve{B}_2,(0.32,0.38),(0.1,0.04)>,$$
$$<\breve{B}_3,(0.1,0.36),(0.34,0.12)>,<\breve{B}_4,(0.26,0.15),(0.12,0.2)>\}.$$

R.H.S:

$$\left(\breve{\delta},\breve{G}_2\right)\cup\left(\breve{\delta},\breve{G}_1\right)=\{<\breve{B}_1,(0.2,0.3),(0.24,0.1)>,$$

$$<\breve{B}_2,(0.32,0.38),(0.1,0.04)>,$$
$$<\breve{B}_3,(0.1,0.36),(0.34,0.12)>,<\breve{B}_4,(0.26,0.15),(0.12,0.2)>\}.$$

From above equation, we meet the required result. Similarly, we can prove the remaining.

6| Conclusion

In this article, the basic fundamentals of refined intuitionistic fuzzy Set (RIFS) i.e. RIF subset, Equal RIFS, Complement of RIFS, Null RIFS and aggregation operators i.e. union, intersection, restricted intersection, extended union, extended intersection and restricted difference of two RIFS is defined. All these fundamentals are explained using an illustrative example. Further extension can be sought through developing similarity measures for comparison purposes.

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Interval Valued Pythagorean Fuzzy Ideals in Semigroups

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Abstract

In this paper, we define the new notion of interval-valued Pythagorean fuzzy ideals in semigroups and established the properties of its with suitable examples. Also, we introduce the concept of interval valued Pythagorean fuzzy sub-semigroup, interval valued Pythagorean fuzzy left (resp. right) ideal, interval valued Pythagorean fuzzy bi-ideal, interval valued Pythagorean fuzzy interior ideal and homomorphism of an interval valued Pythagorean fuzzy ideal in semigroups with suitable illustration. We show that every interval valued Pythagorean fuzzy left (resp. right) ideal is an interval valued Pythagorean fuzzy bi-ideal.

Keywords: Pythagorean fuzzy, Fuzzy ideals, Homomorphism, Semigroups.

1 | Introduction

In 1965, Zadeh [18] and [19] introduced the concept of a fuzzy set. He also developed the notion of interval-valued fuzzy set in 1975, which extends the fuzzy set. A semigroup is an algebraic structure comprising a non-empty set together with an associative binary operation. Atanassov [2] introduced the intuitionistic fuzzy set with some properties. Atanassov [3] developed the concept of interval-valued intuitionistic fuzzy set. Thillaigovindan and Chinnadurai [15, 16] discussed interval-valued fuzzy ideals in algebraic structures. In 2018, Chen [4] and [5] introduced the concept of interval-valued Pythagorean fuzzy outranking of various methods in the application. Garg [8] and [9] presented the notion of interval-valued Pythagorean fuzzy sets of multi-criteria decision-making methods. In 2013, Yager [17] started the notion of Pythagorean fuzzy set, the sum of the squares of membership and non-membership belongs to the unit interval [0, 1]. Peng [13] developed the new operations for an interval-valued Pythagorean fuzzy set. Peng and Yang [14] presented the notion of interval-valued Pythagorean fuzzy set.

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In 2019, Hussain et al. [10] started the notions of rough Pythagorean fuzzy ideals in the semigroups. Akram[1] established the properties of fuzzy lie algebras. Kumar et al. [11] approached transportation decision making problems using Pythagorean fuzzy set. Das and Edalatpanah [6] studied the concept of fuzzy linear fractional progress with trapezoidal fuzzy numbers. Edalatpanah [7] used triangular intuitionistic fuzzy numbers to deal with data envelopment analysis model. Najafi and Edalatpanah [12] used iterative methods to study linear complementarily problems. In this paper, we discuss some of the properties of interval-valued Pythagorean fuzzy ideals in the semigroups.

2| Preliminaries

Definition 1. [12]. Let X be a universe of discourse, A Pythagorean Fuzzy Set (PFS) $P = \left\{w, \phi_p(w), \psi_p(w)/w \in X\right\}$ where $\phi: X \to [0,1]$ and $\psi: X \to [0,1]$ represent the degree of membership and non-membership of the object $w \in X$ to the set P subset to the condition $0 \le \left(\phi_p(w)\right)^2 + \left(\psi_p(w)\right)^2 \le 1$ for all $w \in X$. For the sake of simplicity a PFS is denoted as $P = (\phi_p(w), \psi_p(w))$.

3| Interval-Valued Pythagorean Fuzzy Ideals in Semigroups

Definition 2. An Interval-Valued Pythagorean Fuzzy Set (IVPFS) $\tilde{P} = \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right]$ on S is known to be an interval-valued Pythagorean fuzzy sub-semigroup of S. If for all $w_1, w_2 \in S$, it holds.

$$\widetilde{\phi_p}(w_1w_2) \ge \min\left\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_2)\right\},$$

$$\widetilde{\psi_p}(w_1w_2) \le \max\left\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_2)\right\}.$$

Example 1. Consider a semigroup $S = \{u, v, w, x, y\}$ with the Cayley Table.

Table 1. Cayley table.

•	u	v	w	x	y
u	u	u	u	u	u
v	u	v	u	x	u
w	u	y	w	w	y
x	u	v	x	x	v
y	u	y	u	w	u

Define an interval-valued Pythagorean fuzzy set(IVPFS) $\tilde{P} = \left[\widetilde{\phi_p}, \widetilde{\psi_p}\right]$ in S as follows.

S	$\left[\widetilde{\phi_p}(w_1), \widetilde{\psi_p}(w_1)\right]$,
u	[0.7,0.8], [0.1,0.2],
v	[0.4,0.6], [0.4,0.5],
w	[0.3,0.5], [0.5,0.6],
x	[0.1,0.2], [0.3,0.5],
y	[0.3,0.5], [0.5,0.6].

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$$\widetilde{\phi_p}(uv) \geq \min\{\widetilde{\phi_p}(u), \widetilde{\phi_p}(v)\}$$

([0.7,0.8],[0.1,0.2]) ≥[0.4,0.6],[0.1,0.2].

$$\widetilde{\psi_p}(uv) \leq \max\{\widetilde{\psi_p}(u), \widetilde{\psi_p}(v)\}.$$

([0.7,0.8],[0.1,0.2])≤[0.7,0.8],[0.4,0.5].

Thus $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ is an Interval-Valued Pythagorean Fuzzy Sub-Semigroup (IVPFSS) of S.

Definition 3. An IVPFS $\tilde{P} = (\widetilde{\phi_p}, \widetilde{\psi_p})$ on semigroup S, is said to be an interval-valued Pythagorean fuzzy left $(\tilde{P}_{LI})$(resp.right$(\tilde{P}_{RI})$) ideal of S. If for all $w_1, w_2 \in S$, it holds.

$$\widetilde{\phi_p}(w_1 w_2) \geq \widetilde{\phi_p}(w_2);$$

$$\widetilde{\psi_p}(w_1 w_2) \leq \widetilde{\psi_p}(w_2) \text{ (resp.right}(\tilde{P}_{RI}));$$

$$\widetilde{\phi_p}(w_1 w_2) \geq \widetilde{\phi_p}(w_1);$$

$$\widetilde{\psi_p}(w_1 w_2) \leq \widetilde{\psi_p}(w_1).$$

Definition 4. An IVPFS $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ on S is called IVPFI $(\tilde{P}_I)$ of S. If for all $w_1, w_2 \in S$, it $\tilde{P}$ is both a left and right IVPFI of S.

$$(w_1 w_2) \geq \max\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_2)\};$$

$$\widetilde{\psi_p}(w_1 w_2) \leq \min\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_2)\}.$$

Definition 5. An IVPFS $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ on S is known to be an interval-Valued Pythagorean Fuzzy Bi-Ideal (IVPFBI) $(\tilde{P}_{BI})$ of S. If for all $a, w_1, w_2 \in S$ and satisfy.

$$\widetilde{\phi_p}(w_1 a w_2) \geq \min\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_2)\};$$

$$\widetilde{\psi_p}(w_1 a w_2) \leq \max\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_2)\}.$$

Example 2. Consider a semigroup $S = \{u, v, w, x, y\}$ with the Cayley Table.

Define an interval-valued Pythagorean fuzzy set $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ in S as follows.

S	$[\widetilde{\phi_p}(w_1), \widetilde{\psi_p}(w_1)]$,
u	[0.8,0.9], [0.1,0.3],
v	[0.3,0.5], [0.7,0.9],
w	[0.4,0.6], [0.6,0.7],
x	[0.3,0.5], [0.7,0.9],
y	[0.7,0.8], [0.4,0.5].

Thus $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ is an interval valued Pythagorean fuzzy bi-ideal of S.

Definition 7. An IVPFS $\tilde{P} = \left\langle\left[\widetilde{\phi_p}, \widetilde{\psi_p}\right]\right\rangle$ on S is known to be an interval-valued Pythagorean fuzzy interior ideal (IVPFII) $(\tilde{P}_{II})$ of S. If for all $a, w_1, w_2 \in S$ and satisfy.

$$\widetilde{\phi_p}(w_1 a w_2) \geq \widetilde{\phi_p}(a);$$

$$\widetilde{\psi_p}(w_1 a w_2) \leq \widetilde{\psi_p}(a).$$

Definition 8. For any non-empty subset N of a semigroup S is defined to be a structure $\chi_N = \left\{w_1, [\tilde{\phi}_{\chi_N}(w_1), \tilde{\psi}_{\chi_N}(w_1)] | w_1 \in S\right\}$ which is briefly denoted by $\chi_N = [\tilde{\phi}_{\chi_N}, \tilde{\psi}_{\chi_N}]$

where, $\tilde{\phi}_{\chi_N}(w_1) = \begin{cases} \tilde{1} \; if \; x \in N \\ \tilde{0} \; otherwise \end{cases}$ $\widetilde{\psi}_{\chi_N}(w_1) = \begin{cases} \tilde{0} \; if \; x \in N \\ \tilde{1} \; otherwise \end{cases}$.

Theorem 1. Let S be a semigroup. Then the following are equivalent.

The intersection of two interval-valued Pythagorean fuzzy sub-semigroup of S, is an interval-valued Pythagorean fuzzy sub-semigroup of S.

The intersection of two interval-valued Pythagorean fuzzy left (resp. right) ideal of S, is IVPFLI (resp. IVPFRI) of S.

Proof. Let $\widetilde{P_1} = \left[\tilde{\phi}_{p_1}, \tilde{\psi}_{p_1}\right]$ and $\widetilde{P_2} = \left[\tilde{\phi}_{p_2}, \tilde{\psi}_{p_2}\right]$ be two interval-valued Pythagorean fuzzy sub-semigroup of S. Let $w_1, w_2 \in S$.

Then,

$$\left(\tilde{\phi}_{p_1} \cap \tilde{\phi}_{p_2}\right)(w_1, w_2) = \min\left\{\tilde{\phi}_{p_1}(w_1, w_2), \tilde{\phi}_{p_2}(w_1, w_2)\right\}$$

$$\geq \min\left\{\min\left\{\tilde{\phi}_{p_1}(w_1), \tilde{\phi}_{p_1}(w_2)\right\}, \min\left\{\tilde{\phi}_{p_2}(w_1), \tilde{\phi}_{p_2}(w_2)\right\}\right\}$$

$$= \min\left\{\min\left\{\tilde{\phi}_{p_1}(w_1), \tilde{\phi}_{p_2}(w_1)\right\}, \min\left\{\tilde{\phi}_{p_1}(w_2), \tilde{\phi}_{p_2}(w_2)\right\}\right\}$$

$$= \min\left\{\tilde{\phi}_{p_1} \cap \tilde{\phi}_{p_2}(w_1), \tilde{\phi}_{p_1} \cap \tilde{\phi}_{p_2}(w_2)\right\};$$

$$\left(\tilde{\psi}_{p_1} \cup \tilde{\psi}_{p_2}\right)(w_1, w_2) = \max\left\{\tilde{\psi}_{p_1}(w_1, w_2), \tilde{\psi}_{p_2}(w_1, w_2)\right\}$$

$$\leq \max\left\{\max\left\{\tilde{\psi}_{p_1}(w_1), \tilde{\psi}_{p_1}(w_2)\right\}, \max\left\{\tilde{\psi}_{p_2}(w_1), \tilde{\psi}_{p_2}(w_2)\right\}\right\}$$

$$= \max\left\{\max\left\{\tilde{\psi}_{p_1}(w_1), \tilde{\psi}_{p_2}(w_1)\right\}, \max\left\{\tilde{\psi}_{p_1}(w_2), \tilde{\psi}_{p_2}(w_2)\right\}\right\}$$

$$= \max\left\{\tilde{\psi}_{p_1} \cup \tilde{\psi}_{p_2}(w_1), \tilde{\psi}_{p_1} \cup \tilde{\psi}_{p_2}(w_2)\right\}.$$

Therefore, $\tilde{P}_1 \cap \tilde{P}_2 = \left\{\left\langle\left(\tilde{\phi}_{p_1} \cap \tilde{\phi}_{p_2}\right), \left(\tilde{\psi}_{p_1} \cup \tilde{\psi}_{p_2}\right)\right\rangle\right\}$.

Interval-valued Pythagorean fuzzy sub-semigroup of S.

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$$\left(\tilde{\phi}_{p_1} \cap \tilde{\phi}_{p_2}\right)(w_1, w_2) = \min\left\{\tilde{\phi}_{p_1}(w_1, w_2), \tilde{\phi}_{p_2}(w_1, w_2)\right\}$$

$$\geq \min\left\{\tilde{\phi}_{p_1}(w_2), \tilde{\phi}_{p_2}(w_2)\right\}$$

$$= \left(\tilde{\phi}_{p_1} \cap \tilde{\phi}_{p_2}\right)(w_2);$$

$$\left(\tilde{\psi}_{p_1} \cup \tilde{\psi}_{p_2}\right)(w_1, w_2) = \max\left\{\tilde{\psi}_{p_1}(w_1, w_2), \tilde{\psi}_{p_2}(w_1, w_2)\right\}$$

$$\leq \max\left\{\tilde{\psi}_{p_1}(w_2), \tilde{\psi}_{p_2}(w_2)\right\}$$

$$= \left(\tilde{\psi}_{p_1} \cup \tilde{\psi}_{p_2}\right)(w_2).$$

Therefore, $\tilde{P}_1 \cap \tilde{P}_2 = \left\{\left\langle\left(\tilde{\phi}_{p_1} \cap \tilde{\phi}_{p_2}\right), \left(\tilde{\psi}_{p_1} \cup \tilde{\psi}_{p_2}\right)\right\rangle\right\}$ is an interval-valued Pythagorean fuzzy left (resp. right) ideal of S.

Theorem 2. An IVPFS $\tilde{P} = \left[\tilde{\phi}_p, \tilde{\psi}_p\right]$ of a semigroup S is an IVPFBI of S, if and only if $\left\langle\left(\phi_p^L, \phi_p^U\right), \left(\psi_p^L, \psi_p^U\right)\right\rangle$ of S.

Proof. Let $\tilde{P} = \left[\tilde{\phi}_p, \tilde{\psi}_p\right]$ be an interval-valued Pythagorean fuzzy bi-ideal of S, for any $w_1, w_2 \in S$.

Then, we have membership

$$\left[\phi_p^L(w_1w_2), \phi_p^U(w_1w_2)\right] = \tilde{\phi}_p(w_1w_2)$$

$$\geq \min\left\{\tilde{\phi}_p(w_1), \tilde{\phi}_p(w_2)\right\}$$

$$= \min\left\{\left[\phi_p^L(w_1), \phi_p^U(w_1)\right], \left[\phi_p^L(w_2), \phi_p^U(w_2)\right]\right\}$$

$$= \min\left\{\left[\phi_p^L(w_1), \phi_p^L(w_2)\right], \left[\phi_p^U(w_1), \phi_p^U(w_2)\right]\right\}.$$

It follows that $\phi_p^L(w_1w_2) \geq min\left\{\phi_p^L(w_1), \phi_p^L(w_2)\right\}$ and $\phi_p^U(w_1w_2) \geq min\left\{\phi_p^U(w_1), \phi_p^U(w_2)\right\}$and non-membership

$$\left[\psi_p^L(w_1w_2), \psi_p^U(w_1w_2)\right] = \tilde{\psi}_p(w_1w_2)$$

$$\leq \max\left\{\tilde{\psi}_p(w_1), \tilde{\psi}_p(w_2)\right\}$$

$$= \max\left\{\left[\psi_p^L(w_1), \psi_p^U(w_1)\right], \left[\psi_p^L(w_2), \psi_p^U(w_2)\right]\right\}$$

$$= \max\left\{\left[\psi_p^L(w_1), \psi_p^L(w_2)\right], \left[\psi_p^U(w_1), \psi_p^U(w_2)\right]\right\}.$$

It follows that $\psi_p^L(w_1w_2) \leq max\left\{\psi_p^L(w_1), \psi_p^L(w_2)\right\}$ and $\psi_p^U(w_1w_2) \leq max\left\{\psi_p^U(w_1), \phi_p^U(w_2)\right\}$

Therefore, $\tilde{P} = \left\langle\left(\phi_p^L, \phi_p^U\right), \left(\psi_p^L, \psi_p^U\right)\right\rangle$ are Pythagorean fuzzy ideal of S.

Conversely, suppose that $\left(\left[\phi_p^L, \phi_p^U\right], \left[\psi_p^L, \psi_p^U\right]\right)$ are Pythagorean fuzzy ideal of S, $le w_1, w_2 \in S$ t.

$$\tilde{\phi}_p(w_1w_2) = \left[\phi_p^L(w_1w_2), \phi_p^U(w_1w_2)\right]$$

$$\geq \left[\min\left\{\phi_p^L(w_1), \phi_p^L(w_2)\right\}, \min\left\{\phi_p^U(w_1), \phi_p^U(w_2)\right\}\right]$$

$$= \min\left\{\left[\phi_p^L(w_1), \phi_p^U(w_1)\right], \left[\phi_p^L(w_2), \phi_p^U(w_2)\right]\right\}$$

$$= \min\left\{\tilde{\phi}_p(w_1), \tilde{\phi}_p(w_2)\right\};$$

$$\tilde{\psi}_p(w_1w_2) = \left[\psi_p^L(w_1w_2), \psi_p^U(w_1w_2)\right]$$

$$\leq \left[\max\left\{\psi_p^L(w_1), \psi_p^L(w_2)\right\}, \max\left\{\psi_p^U(w_1), \psi_p^U(w_2)\right\}\right]$$

$$= \max\left\{\left[\psi_p^L(w_1), \psi_p^U(w_1)\right], \left[\psi_p^L(w_2), \psi_p^U(w_2)\right]\right\}$$

$$= \max\left\{\tilde{\psi}_p(w_1), \tilde{\psi}_p(w_2)\right\}.$$

$\tilde{P} = \left[\tilde{\phi}_p, \tilde{\psi}_p\right]$ is an interval-valued Pythagorean fuzzy sub-semigroup of S.

$$\tilde{\phi}_p(w_1aw_2) = \left[\phi_p^L(w_1aw_2), \phi_p^U(w_1aw_2)\right]$$

$$\geq \left[\min\left\{\phi_p^L(w_1), \phi_p^L(w_2)\right\}, \min\left\{\phi_p^U(w_1), \phi_p^U(w_2)\right\}\right]$$

$$= \min\left\{\left[\phi_p^L(w_1), \phi_p^U(w_1)\right], \left[\phi_p^L(w_2), \phi_p^U(w_2)\right]\right\}$$

$$= \min\left\{\tilde{\phi}_p(w_1), \tilde{\phi}_p(w_2)\right\};$$

$$\tilde{\psi}_p(w_1aw_2) = \left[\psi_p^L(w_1aw_2), \psi_p^U(w_1aw_2)\right]$$

$$\leq \left[\max\left\{\psi_p^L(w_1), \psi_p^L(w_2)\right\}, \max\left\{\psi_p^U(w_1), \psi_p^U(w_2)\right\}\right]$$

$$= \max\left\{\left[\psi_p^L(w_1), \psi_p^U(w_1)\right], \left[\psi_p^L(w_2), \psi_p^U(w_2)\right]\right\}$$

$$= \max\left\{\tilde{\psi}_p(w_1), \tilde{\psi}_p(w_2)\right\}.$$

$\tilde{P} = \left[\tilde{\phi}_p, \tilde{\psi}_p\right]$ is an interval-valued Pythagorean fuzzy bi-ideal of S.

Theorem 3. If $\{P_i\}_{i\in I}$ is a family of interval-valued Pythagorean fuzzy bi-ideal of a semigroup S. Then $\cap P_i$ is an interval-valued Pythagorean fuzzy bi-ideal of S. Where $\cap P_i = \left(\cap \tilde{\phi}_{p_i}, \cup \tilde{\psi}_{p_i}\right)$.

$\cap\left(\tilde{\phi}_{p_i}\right) = inf\left\{\left(\tilde{\phi}_{p_i}\right)(w_1)/i \in I, w_1 \in S\right\}, \cup\left(\tilde{\psi}_{p_i}\right) = sup\left\{\left(\tilde{\psi}_{p_i}\right)(w_1)/i \in I, w_1 \in S\right\}$ *and* $i \in I$ *is* any index set.

Proof. Since $\tilde{P}_i = \left\langle\left[\tilde{\phi}_{p_i}, \tilde{\psi}_{p_i}\right] | i \in I\right\rangle$ is a family of interval-valued Pythagorean fuzzy bi-ideal of S.

Let $a, w_1, w_2 \in S$.

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$$\cap \tilde{\phi}_{p_i}(w_1, w_2) = \inf\{\tilde{\phi}_{p_i}(w_1, w_2)/i \in I, w_1, w_2 \in S\}$$

$$\geq \inf\left\{\min\{\tilde{\phi}_{p_i}(w_1), \tilde{\phi}_{p_i}(w_2)\}\right\}$$

$$= \min\left\{\inf\left(\tilde{\phi}_{p_i}(w_1)\right), \inf\left(\tilde{\phi}_{p_i}(w_2)\right)\right\}$$

$$= \min\{\cap \tilde{\phi}_{p_i}(w_1), \cap \tilde{\phi}_{p_i}(w_2)\};$$

$$\cup \tilde{\psi}_{p_i}(w_1 w_2) = \sup\{\tilde{\psi}_{p_i}(w_1 w_2)/i \in I, w_1, w_2 \in S\}$$

$$\leq \sup\left\{\max\{\tilde{\psi}_{p_i}(w_1), \tilde{\psi}_{p_i}(w_2)\}\right\}$$

$$= \max\left\{\sup\left(\tilde{\psi}_{p_i}(w_1)\right), \sup\left(\tilde{\psi}_{p_i}(w_2)\right)\right\}$$

$$= \max\{\cup \tilde{\psi}_{p_i}(w_1), \cup \tilde{\psi}_{p_i}(w_2)\}.$$

Hence, $\cap\ \tilde{P}_i = \left(\cap \tilde{\phi}_{p_i}, \cup \tilde{\psi}_{p_i}\right)$ is an interval-valued Pythagorean fuzzy sub-semigorup of S.

$$\cap \tilde{\phi}_{p_i}(w_1 a w_2) = \inf\{\tilde{\phi}_{p_i}(w_1 a w_2)/i \in I, a, w_1, w_2 \in S\}$$

$$\geq \inf\left\{\min\{\tilde{\phi}_{p_i}(w_1), \tilde{\phi}_{p_i}(w_2)\}\right\}$$

$$= \min\left\{\inf\left(\tilde{\phi}_{p_i}(w_1)\right), \inf\left(\tilde{\phi}_{p_i}(w_2)\right)\right\}$$

$$= \min\{\cap \tilde{\phi}_{p_i}(w_1), \cap \tilde{\phi}_{p_i}(w_2)\}.$$

$$\cup \tilde{\psi}_{p_i}(w_1 a w_2) = \sup\{\tilde{\psi}_{p_i}(w_1 a w_2)/i \in I, a, w_1, w_2 \in S\}$$

$$\leq \sup\left\{\max\{\tilde{\psi}_{p_i}(w_1), \tilde{\psi}_{p_i}(w_2)\}\right\}$$

$$= \max\left\{\sup\left(\tilde{\psi}_{p_i}(w_1)\right), \sup\left(\tilde{\psi}_{p_i}(w_2)\right)\right\}$$

$$= \max\{\cup \tilde{\psi}_{p_i}(w_1), \cup \tilde{\psi}_{p_i}(w_2)\}.$$

Hence, $\cap P_i = \left(\cap \tilde{\phi}_{p_i}, \cup \tilde{\psi}_{p_i}\right)$ is an interval-valued Pythagorean fuzzy bi-ideals of S.

Theorem 4. Let N be any non-empty subset of a semigroup S. Then N is a bi-ideal of S, if and only if the characteristic interval-valued Pythagorean fuzzy set $\chi_N = \left[\tilde{\phi}_{p\chi_N}, \tilde{\psi}_{p\chi_N}\right]$ is IVPFBI of S.

Proof. Assume that N is a bi-ideal of S. Let $a, w_1, w_2 \in S$.

Suppose that $\tilde{\phi}_{p\chi_N}(w_1 w_2) < min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\}$ and $\tilde{\psi}_{p\chi_N}(w_1 w_2) > max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\}$ it follows

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that $\tilde{\phi}_{p\chi_N}(w_1w_2) = 0,\ min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\} = 1, \tilde{\psi}_{p\chi_N}(w_1w_2) = 1,\ max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\} = 0.$

This implies that $w_1, w_2 \in N$ by $w_1, w_2 \notin N$ a contradiction to N.

So $\tilde{\phi}_{p\chi_N}(w_1w_2) \geq min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\}$, $\tilde{\psi}_{p\chi_N}(w_1w_2) \leq max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\}$.

Suppose that $\tilde{\phi}_{p\chi_N}(w_1aw_2) < min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\}$ and $\tilde{\psi}_{p\chi_N}(w_1aw_2) > max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\}$ it follows that $\tilde{\phi}_{p\chi_N}(w_1aw_2) = 0,\ min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\} = 1, \tilde{\psi}_{p\chi_N}(w_1w_2) = 1,$
$max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\} = 0.$

This implies that $a, w_1, w_2 \in N$ by $a, w_1, w_2 \notin N$ a contradiction to N.

So $\tilde{\phi}_{p\chi_N}(w_1aw_2) \geq min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\}$, $\tilde{\psi}_{p\chi_N}(w_1aw_2) \leq max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\}$.

This shows that χ_N is an interval-valued Pythagorean fuzzy bi-ideal of S.

Conversely, $\chi_N = [\tilde{\phi}_{p\chi_N}, \tilde{\psi}_{p\chi_N}]$ is an IVPFBI of S for any subset N of S.

Let $w_1, w_2 \in N$ then $\tilde{\phi}_{p\chi_N}(w_1) = \tilde{\phi}_{p\chi_N}(w_2) = \tilde{1}$, $\tilde{\psi}_{p\chi_N}(w_1) = \tilde{\psi}_{p\chi_N}(w_2) = \tilde{0}$, since χ_N is an IVPFBI of S.

$\tilde{\phi}_{p\chi_N}(w_1w_2) \geq min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\} \geq min\{\tilde{1}, \tilde{1}\} = \tilde{1}, \tilde{\psi}_{p\chi_N}(w_1w_2) \leq max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\} \leq max\{\tilde{0}, \tilde{0}\} = \tilde{0}.$

This implies that $w_{1,}w_2 \in N$.

Let $a, w_1, w_2 \in N$ then $\tilde{\phi}_{p\chi_N}(w_1) = \tilde{\phi}_{p\chi_N}(a) = \tilde{\phi}_{p\chi_N}(w_2) = \tilde{1}$,

$\tilde{\psi}_{p\chi_N}(w_1) = \tilde{\psi}_{p\chi_N}(a) = \tilde{\psi}_{p\chi_N}(w_2) = \tilde{0}$, since χ_N is an IVPFBI of S.

$\tilde{\phi}_{p\chi_N}(w_1aw_2) \geq min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\} \geq min\{\tilde{1}, \tilde{1}\} = \tilde{1}, \tilde{\psi}_{p\chi_N}(w_1aw_2) \leq max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\} \leq max\{\tilde{0}, \tilde{0}\} = \tilde{0}.$

Which implies that $w_1, w_2 \in N$. Hence N is a bi- ideal of S.

Theorem 5. If $\{\tilde{P}_i\}_{i\in I}$ is a family of interval-valued Pythagorean fuzzy interior ideal of a semigroup S. Then $\cap\ \tilde{P}_i$ is an interval-valued Pythagorean fuzzy interior ideal (IVPFII) of S.

Where $\cap\ \tilde{P}_i = (\cap\, \tilde{\phi}_{p_i}, \cup\, \tilde{\psi}_{p_i})$;

$\cap\,(\tilde{\phi}_{p_i}) = inf\{(\tilde{\phi}_{p_i})(w_1)/i \in I, w_1 \in S\}, \cup\,(\tilde{\psi}_{p_i}) = sup\{(\tilde{\psi}_{p_i})(w_1)/i \in I, w_1 \in S\}$ and $i \in I$ is any index set.

Theorem 6. Let N be any non-empty subset of a semigroup S. Then N is a interior ideal of S, if and only if the characteristic interval-valued Pythagorean fuzzy set $\chi_N = [\tilde{\phi}_{p\chi_N}, \tilde{\psi}_{p\chi_N}]$ is IVPFII of S.

4| Homomorphism of Interval-Valued Pythagorean Fuzzy Ideals in Semigroups

Let R and T be two non-empty sets of semigroup S. A mapping $f: R \rightarrow T$ is called a homomorphism if $(rt) = f(r)f(t)\ \forall\, r, t \in R$.

Definition 9. Let f be a mapping from a set R to a set T and $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ be an interval-valued Pythagorean fuzzy set R the image of R (i.e.) $f(\tilde{P}) = (f(\tilde{\phi}_p), f(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy set of T is defined by

$$f(\tilde{P})(r) = \begin{cases} f(\widetilde{\phi_P})(r) = \begin{cases} \sup\limits_{t \in f'(r)} (\widetilde{\phi_P})(t), & \text{if } f^{-1}(r) = 0 \\ [0,0] & \text{otherwise} \end{cases} \\ f(\widetilde{\psi_P})(r) = \begin{cases} \inf\limits_{t \in f'(r)} (\widetilde{\psi_P})(t), & \text{if } f^{-1}(r) = 0 \\ [1,1] & \text{otherwise} \end{cases} \end{cases}$$

Let f be a mapping from a set R to T and $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ be an interval-valued Pythagorean fuzzy set of T then the preimage of T (i.e.) $f^{-1}(\tilde{P}) = \{(f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))\}$ is an interval-valued Pythagorean fuzzy set of R is defined as

$$f^{-1}(\tilde{P})(r) = \begin{cases} f^{-1}(\widetilde{\phi_p})(r) = \widetilde{\phi_p}(f(r)) \\ f^{-1}(\widetilde{\psi_p})(r) = \widetilde{\psi_p}(f(r)) \end{cases}.$$

Theorem 7. Let R, T be a semigroups, $f: R \to T$ be a homomorphism of semigroups.

If $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy sub-semigroup of T the the preimage $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy sub-semigroup of R.

If $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy left (resp.right) ideal of T the the preimage $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy left ideal (resp. right ideal) of R.

Proof. Assume that $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy sub-semigroup of T and $r, t \in R$. Then

$$f^{-1}(\tilde{\phi}_p)(rt) = \tilde{\phi}_p(f(rt))$$

$$= \tilde{\phi}_p(f(r)f(t))$$

$$\geq \min\{\tilde{\phi}_p(f(r)), \tilde{\phi}_p(f(t))\}$$

$$= \min\{f^{-1}(\tilde{\phi}_p)(r), f^{-1}(\tilde{\phi}_p)(f(t))\};$$

$$f^{-1}(\tilde{\psi}_p)(rt) = \tilde{\psi}_p(f(rt))$$

$$= \tilde{\psi}_p(f(r)f(t))$$

$$\leq \max\{\tilde{\psi}_p(f(r)), \tilde{\psi}_p(f(t))\}$$

$$= \max\{f^{-1}(\tilde{\psi}_p)(r), f^{-1}(\tilde{\psi}_p)(f(t))\}.$$

Hence, $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy sub-semigroup of R.

$$f^{-1}(\tilde{\phi}_p)(rt) = \tilde{\phi}_p(f(rt))$$

$$= \tilde{\phi}_p(f(r)f(t))$$

$$\geq \tilde{\phi}_p(f(t))$$

$$= f^{-1}\left(\tilde{\phi}_p\right)\left(f(t)\right);$$

$$f^{-1}(\tilde{\psi}_p)(rt) = \tilde{\psi}_p(f(rt))$$

$$= \tilde{\psi}_p(f(r)f(t))$$

$$\leq \tilde{\psi}_p(f(t))$$

$$= f^{-1}\left(\tilde{\psi}_p\right)\left(f(t)\right).$$

Hence, $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy left (resp.right) ideal of R.

Theorem 8. Let R, T be a semigroups, $f: R \to T$ be a homomorphism of semigroups. If $\tilde{P} = \left[\tilde{\phi}_p, \tilde{\psi}_p\right]$ is an interval-valued Pythagorean fuzzy bi-ideal of T the the preimage $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy bi-ideal of R.

Proof. Assume that $\tilde{P} = \left[\tilde{\phi}_p, \tilde{\psi}_p\right]$ is an interval-valued Pythagorean fuzzy sub-semigroup of T and $a, r, t \in R$. Then

$$f^{-1}(\tilde{\phi}_p)(rat) = \tilde{\phi}_p(f(rat))$$

$$= \tilde{\phi}_p(f(r)f(a)f(t))$$

$$\geq \min\left\{\tilde{\phi}_p(f(r)), \tilde{\phi}_p(f(t))\right\}$$

$$= \min\left\{f^{-1}(\tilde{\phi}_p)(r), f^{-1}(\tilde{\phi}_p)(f(t))\right\};$$

$$f^{-1}(\tilde{\psi}_p)(rat) = \tilde{\psi}_p(f(rat))$$

$$= \tilde{\psi}_p(f(r)f(a)f(t))$$

$$\leq \max\left\{\tilde{\psi}_p(f(r)), \tilde{\psi}_p(f(t))\right\}$$

$$= \max\left\{f^{-1}(\tilde{\psi}_p)(r), f^{-1}(\tilde{\psi}_p)(f(t))\right\}.$$

Hence $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy bi-ideal of R.

Theorem 9. Let R, T be a semigroups, $f: R \to T$ be a homomorphism of semigroups. If $\tilde{P} = \left[\tilde{\phi}_p, \tilde{\psi}_p\right]$ is an interval-valued Pythagorean fuzzy interior ideal of T the preimage $f^{-1}(P) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy interior ideal of R.

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5| Conclusion

In this paper interval valued Pythagorean fuzzy sub-semigroup, interval valued Pythagorean fuzzy left (resp. right) ideal, interval valued Pythagorean fuzzy ideal, interval valued Pythagorean fuzzy bi-ideal, interval valued Pythagorean fuzzy interior ideal and Homomorphism of interval valued Pythagorean fuzzy ideal in semigroups are studied and investigated some properties with suitable examples.

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Some Similarity Measures of Rough Interval Pythagorean Fuzzy Sets

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Abstract

The purpose of this study is to propose new similarity measures namely cosine, jaccard and dice similarity measures. The weighted cosine, weighted jaccard and weighted dice similarity measures has been also defined. Some of the important properties of the defined similarity measures and weighted similarity measures have been established. We develop a new multi attribute decision making problem based on the proposed similarity measures. To demonstrate the applicability, a numerical example is solved.

Keywords: Interval valued fuzzy set, Pythagorean fuzzy set, Rough set, Cosine similarity measure, Jaccard similarity measure, Dice similarity measure.

1 | Introduction

The concept of fuzzy set was introduced by Zadeh [15] in his classic paper in 1965 and has been applied to many branches in mathematics. Later Zadeh [14] also introduced the concept of interval valued fuzzy set by considering the values of membership functions as the intervals of numbers instead of the numbers alone. The notion of rough set theory was proposed by Pawlak [7]. The concept of rough set theory is an extension of crisp set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Dubois and Prade [3] were introduced the concept of rough fuzzy set. This theory was found to be more useful in decision making and medical diagnosis problems. A similarity measure is an important tool for determining the degree of similarity between two objects. Similarity measures between fuzzy sets is an important content in fuzzy mathematics. Yager [13] examined Pythagorean fuzzy set characterized by a membership degree and a non-membership degree that satisfies the case in which the square sum of its membership degree and non-membership degree is less than or equal to one.

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Peng and Yang [9] introduced the concept of interval Pythagorean fuzzy sets which is a generalization of Pythagorean fuzzy sets and interval valued fuzzy sets. Hussain et al. [2] introduced the concept of rough Pythagorean fuzzy sets. The Pythagorean fuzzy set has been investigated from different perspectives, including decision-making technologies [8], medical diagnosis [10], and transportation problem [6]. In particular, an extension of Pythagorean fuzzy set, named Interval-Valued Pythagorean Fuzzy Sets in decision making [8], complex Pythagoren fuzzy set in pattern recognition [12].

To facilitate our discussion, the remainder of this paper is organized as follows. In Section 2 we review some fundamental conceptions rough sets, interval valued fuzzy sets, Pythgorean fuzzy sets. In Section 3 we propose cosine similarity measure of rough interval Pythagorean fuzzy sets and some properties of this similarity measure discussed. Sections 4 and 5 deals with jaccard, dice similarity measures. In Section 6 we present algorithm for proposed measures. Section 7 deals with numerical example of proposed measures.

2| Basic Concepts

In this section we list some basic concepts.

Definition 1. Let x be a nonempty set. A mapping $\tilde{\Omega}: x \to D[0,1]$ is called an interval valued fuzzy subset of x, where $\tilde{\Omega}(x) = [\Omega^-(x), \Omega^+(x)], x \in X$, and Ω^- and Ω^+ are the fuzzy subets in X such that $\Omega^-(x) \le \Omega^+(x)$ $x \in X$. $D[0,1]$ denotes the set of closed subsets of [0,1].

Definition 2. [5]. Let ϑ be a congruence relation on X. Le Λ t be any nonempty subset of X. The sets $\underline{\vartheta}(\Lambda) = \{x \in X/[x]_\vartheta \subseteq \Lambda\}$ *and* $\overline{\vartheta}(\Lambda) = \{x \in X/[x]_\vartheta \cap \vartheta \ne \emptyset\}$ are called the lower and upper approximations of Λ. Then $\vartheta(\Lambda) = (\underline{\vartheta}(\Lambda), \overline{\vartheta}(\Lambda))$ is called rough set in $(X, \vartheta) \Longleftrightarrow \underline{\vartheta}(\Lambda) \ne \overline{\vartheta}(\Lambda)$.

Definition 3. [3]. Let ϑ be an congruence relation on X. Let Λ fuzzy subset of X. The upper and lower approximations of Λ defined by $\overline{\vartheta}(\Lambda)(x) = \bigvee_{a \in [x]_\vartheta} \Lambda(a)$ and $\underline{\vartheta}(\Lambda)(x) = \bigwedge_{a \in [x]_\vartheta} \Lambda(a)$. $\vartheta(\Lambda) = (\underline{\vartheta}(\Lambda), \overline{\vartheta}(\Lambda))$ is called a rough fuzzy set of Λ with respect to ϑ if $\underline{\vartheta}(\Lambda) \ne \overline{\vartheta}(\Lambda)$.

Definition 4. [4]. Let $\tilde{\Omega}$ be an interval-valued fuzzy subset of X and let ϑ be the complete congruence relation on X. Let $\underline{\vartheta}(\tilde{\Omega})$ and $\overline{\vartheta}(\tilde{\Omega})$ be the interval-valued fuzzy subset of X defined by, $\underline{\vartheta}(\tilde{\Omega})(n) = \wedge_{n \in [y]_\vartheta} \tilde{\Omega}(n)$ and $\overline{\vartheta}(\tilde{\Omega})(n) = \vee_{n \in [y]_\vartheta} \tilde{\Omega}(n)$. Then $\vartheta(\tilde{\Omega}) = \left(\underline{\vartheta}(\tilde{\Omega}), \overline{\vartheta}(\tilde{\Omega})\right)$ is called an interval-valued rough fuzzy subset of X if $\underline{\vartheta}(\tilde{\Omega}) \ne \overline{\vartheta}(\tilde{\Omega})$.

Definition 5. [1]. Let X be a nonempty set then an Intutionistic fuzzy set can be defined as $\Lambda\Omega = \left\{\left(x, \mu_\Omega(x), \gamma_{\Omega_\Lambda}(x)\right)/x \in X\right\}$ where $\mu_{\Omega_\Lambda}(x)$ *and* $\gamma_\Lambda(x)$ are mapping from X to [0,1] also $0 \le \mu_{\Omega_\Lambda}(x) \le 1, 0 \le \gamma_{\Omega_{\Lambda_\Lambda}}(x) \le 1, 0 \le \mu_{\Omega_\Lambda}(x) + \gamma_{\Omega_\Lambda}(x) \le 1$ for all $x \in X$ and represent the degrees of membership and non-membership of element $x \in X$ to set X.

Definition 6. [11]. Let X be a nonempty set then an Pythagorean fuzzy set can be defined as $\Omega = \left\{\left(x, \mu_\Omega(x), \gamma_{\Omega_\Lambda}(x)\right)/x \in X\right\}$ where $\mu_\Omega(x)$ *and* $\gamma_{\Omega_\Lambda}(x)$ are mapping from X to [0,1] also $0 \le \mu_{\Omega_\Lambda}(x) \le 1, 0 \le \gamma_\Omega(x) \le 1, 0 \le {\mu_\Omega}^2{}_\Lambda(x) + {\gamma_\Omega}^2{}_\Lambda(x) \le 1$ for all $x \in X$, and represent the degrees of membership and non membership of element $x \in X$ to set X.

Definition 7. [7]. Let X be a non-empty set then an Interval Pythagorean fuzzy set can be defined as follows $\tilde{\Omega} = \left\{\left(x, \mu_{\tilde{\Omega}}(x), \gamma_{\tilde{\Omega}}(x)\right)/x \in X\right\}$ where $\mu_{\tilde{\Omega}}(x) = [{\mu_{\tilde{\Omega}}}^-(x), {\mu_{\tilde{\Omega}}}^+(x)]$ and $\gamma_{\tilde{\Omega}}(x) = [{\gamma_{\tilde{\Omega}}}^-(x), {\gamma_{\tilde{\Omega}}}^+(x)]$ are the intervals in [0,1] also $0 \le (\mu^+{}_{\tilde{\Omega}}(x))^2 + (\gamma^+{}_{\tilde{\Omega}}(x))^2 \le 1$.

3| Cosine Similarity Measures (CSM) of Rough Interval Pythagorean Fuzzy (RIPF) Sets.

In this section we introduce the notion of CSM of *RIPF* sets also discuss some properties of *RIPF* sets. Also weighted CSM of *RIPF* sets are discussed.

Definition 8. Let X be a nonempty set. Let $\tilde{\Omega} = \left\{\left(n, \mu_{\tilde{\Omega}}(n), \gamma_{\tilde{\Omega}}(n)\right)/n \in X\right\}$ be a pythagorean fuzzy set of X. Then rough interval Pythagorean fuzzy set is defined as $\vartheta(\tilde{\Omega}) = (\underline{\vartheta}(\tilde{\Omega}), \overline{\vartheta}(\tilde{\Omega}))$ where

$$\underline{\vartheta}(\tilde{\Omega}) = \left\{\langle n, \underline{\vartheta}(\mu_{\tilde{\Omega}}), \underline{\vartheta}(\gamma_{\tilde{\Omega}})\rangle, n \in X\right\} \text{ and } \overline{\vartheta}\left(\tilde{\Omega}\right) = \left\{\langle n, \overline{\vartheta}(\mu_{\tilde{\Omega}}), \overline{\vartheta}(\gamma_{\tilde{\Omega}})\rangle, n \in X\right\},$$

with the condition that $0 \le \left(\underline{\vartheta}(\mu_{\tilde{\Omega}})\right)^2 + \left(\underline{\vartheta}(\gamma_{\tilde{\Omega}})\right)^2 \le 1$, $0 \le \left(\overline{\vartheta}(\mu_{\tilde{\Omega}})\right)^2 + \left(\overline{\vartheta}(\gamma_{\tilde{\Omega}})\right)^2 \le 1$.

Here, $\underline{\vartheta}(\mu_{\tilde{\Omega}})(n) = \wedge_{n\in[y]_\vartheta} \mu_{\tilde{\Omega}}(y)$ and $\underline{\vartheta}(\gamma_{\tilde{\Omega}})(n) = \vee_{n\in[y]_\vartheta} \gamma_{\tilde{\Omega}}(y)$ also,

$\overline{\vartheta}(\mu_{\tilde{\Omega}})(n) = \vee_{n\in[y]_\vartheta} \mu_{\tilde{\Omega}}(y)$ and $\overline{\vartheta}(\gamma_{\tilde{\Omega}})(n) = \wedge_{n\in[y]_\vartheta} \gamma_{\tilde{\Omega}}(y)$.

Definition 9. Let ϑ be an congruence relation on X. Consider two *RIPF* sets $\vartheta\left(\widetilde{\Omega_1}\right)$, $\vartheta\left(\widetilde{\Omega_2}\right)$ in $X = \{x_1, x_2 \dots \dots x_n\}$. A *CSM* between $\vartheta\left(\widetilde{\Omega_1}\right)$ and $\vartheta\left(\widetilde{\Omega_2}\right)$ is defined as follows:

$$C_{RIPF}\left(\vartheta\left(\widetilde{\Omega_1}\right), \vartheta\left(\widetilde{\Omega_2}\right)\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2 + (\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i))^2}\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i))^2 + (\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i))^2}}. \tag{1}$$

Where

$$\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{\left(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i))\right)}{4};$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{\left(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i))\right)}{4};$$

$$\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{\left(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i))\right)}{4};$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{\left(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i))\right)}{4}.$$

Proposition 1. A *RIPCSM* between $\vartheta\left(\widetilde{\Omega_1}\right)$ and $\vartheta\left(\widetilde{\Omega_2}\right)$ satisfies the following properties:

$$0 \le C_{RIPF}\left(\vartheta\left(\widetilde{\Omega_1}\right), \vartheta\left(\widetilde{\Omega_2}\right)\right) \le 1;$$

$$C_{RIPF}\left(\vartheta\left(\widetilde{\Omega_1}\right), \vartheta\left(\widetilde{\Omega_2}\right)\right) = 1 \Longleftrightarrow \vartheta\left(\widetilde{\Omega_1}\right) = \vartheta\left(\widetilde{\Omega_2}\right);$$

$$C_{RIPF}\left(\vartheta\left(\widetilde{\Omega_1}\right), \vartheta\left(\widetilde{\Omega_2}\right)\right) = C_{RIPF}\left(\vartheta\left(\widetilde{\Omega_2}\right), \vartheta\left(\widetilde{\Omega_1}\right)\right).$$

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Proof. It is obvious because all positive values of cosine function are within 0 and 1; it is obvious; for any two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$, if $\vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2})$ then,

$\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)$ and $\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)$. Hence $cos(0) = 1$. Conversely, if $C_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1$, then $\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)$ and $\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)$. Hence $\vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2})$.

If we consider weight ω_i of each elementx_i, a weighted *RICSM* between *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ is defined as follows:

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2 + (\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i))^2}\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i))^2 + (\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i))^2}}. \quad (2)$$

$\omega_i \in [0,1], i = 1,2,3 \dots n$ and $\sum_{i=1}^{n}\omega_i = 1$. If we take $\omega_i = \frac{1}{n}, i = 1,2, \dots . n$ then

$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = C_{RIPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right)$.

The weighted *RICSM* between two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ also satisfies the following properties.

Proposition 2.

$$0 \le C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \le 1;$$

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1 \Longleftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2});$$

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = C_{WRIPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right).$$

4| Jaccard Similarity Measure (JSM) of Rough Interval Pythagorean Fuzzy (RIPF) Set

In this section we introduce the concept of *JSM* of *RIPF* sets. Weighted *JSM* of *RIPF* also derived.

Definition 10. Let ϑ be an congruence relation on *X*. Consider two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$, $\vartheta(\widetilde{\Omega_2})$ in $X = \{x_1, x_2 \dots \dots x_n\}$. A *JSM* between $\vartheta(\widetilde{\Omega_1})$and $\vartheta(\widetilde{\Omega_2})$ is defined as follows:

$$J_{IRPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\begin{array}{c}[(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2 + \left(\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\right)^2 + \left(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)^2 + \left(\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)^2 + \\ \delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)]\end{array}}, \quad (3)$$

where

$\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{\left(\underline{\vartheta}(\mu^-(x_i))+\underline{\vartheta}(\mu^+(x_i))+\overline{\vartheta}(\mu^-(x_i))+\overline{\vartheta}(\mu^+(x_i))\right)}{4}$,

$\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{\left(\underline{\vartheta}(\gamma^-(x_i))+\underline{\vartheta}(\gamma^+(x_i))+\overline{\vartheta}(\gamma^-(x_i))+\overline{\vartheta}(\gamma^+(x_i))\right)}{4}$ and

$$\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{\left(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i))\right)}{4},$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{\left(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i))\right)}{4}.$$

Proposition 3. A RIPJSM between $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ satisfies the following properties:

$$0 \le J_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \le 1,$$

$$J_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1 \Longleftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2}),$$

$$J_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = J_{IRPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right).$$

Proof. It is obvious because all positive values of cosine function are within 0 and 1; it is obvious; for any two RIPF sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$, if $\vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2})$ then, $\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)$ and $\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)$. Hence $cos(0) = 1$. Conversely, if $J_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1$, then $\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)$ and $\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)$. Hence $\vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2})$.

If we consider weight ω_i of each element x_i, a weighted RIPJSM between RIPF sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ is defined as follows:

$$J_{IRPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \frac{1}{n}\sum_{i=1}^{n} \omega_i \frac{\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\begin{gathered}[(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2 + \left(\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\right)^2 + \left(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)^2 + \left(\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)^2 + \\ \delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)]\end{gathered}}. \quad (4)$$

$\omega_i \in [0,1], i = 1,2,3 \ldots n$ and $\sum_{i=1}^{n} \omega_i = 1$. If we take $\omega_i = \frac{1}{n}, i = 1,2, \ldots . n$ then $J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = J_{RIPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right)$.

The weighted RIPJSM between two RIPF sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ also satisfies the following properties.

Proposition 4.

$$0 \le J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \le 1;$$

$$J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1 \Longleftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2});$$

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$$\mathrm{J}_{\mathrm{WRIPF}}\left(\vartheta\left(\widetilde{\Omega_1}\right),\vartheta\left(\widetilde{\Omega_2}\right)\right)=\mathrm{J}_{\mathrm{WRIPF}}\left(\vartheta\left(\widetilde{\Omega_2}\right),\vartheta\left(\widetilde{\Omega_1}\right)\right).$$

5| Dice Similarity Measure (DSM) of Rough Interval Pythagorean Fuzzy (RIPF) Set

This section deals with DSM of *RIPF* sets. Some properties of this similarity measure are discussed.

Definition 11. Let $\vartheta\left(\widetilde{\Omega_1}\right)$ and $\vartheta\left(\widetilde{\Omega_2}\right)$ be two *RIPF* set in$X=\{x_1,x_2 \ldots \ldots x_n\}$. A *DSM* between $\vartheta\left(\widetilde{\Omega_1}\right)$and $\vartheta\left(\widetilde{\Omega_2}\right)$ is defined as follows:

$$\mathrm{D}_{\mathrm{RIPF}}\left(\vartheta(\widetilde{\Omega_1}),\vartheta(\widetilde{\Omega_2})\right)=\frac{1}{n}\sum_{i=1}^{n}\frac{2\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)+\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2+(\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i))^2}\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i))^2+(\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i))^2}}. \quad (5)$$

Where

$$\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)=\frac{\left(\underline{\vartheta}(\mu^-(x_i))+\underline{\vartheta}(\mu^+(x_i))+\overline{\vartheta}(\mu^-(x_i))+\overline{\vartheta}(\mu^+(x_i))\right)}{4},$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)=\frac{\left(\underline{\vartheta}(\gamma^-(x_i))+\underline{\vartheta}(\gamma^+(x_i))+\overline{\vartheta}(\gamma^-(x_i))+\overline{\vartheta}(\gamma^+(x_i))\right)}{4} \text{ and}$$

$$\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)=\frac{\left(\underline{\vartheta}(\mu^-(x_i))+\underline{\vartheta}(\mu^+(x_i))+\overline{\vartheta}(\mu^-(x_i))+\overline{\vartheta}(\mu^+(x_i))\right)}{4},$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)=\frac{\left(\underline{\vartheta}(\gamma^-(x_i))+\underline{\vartheta}(\gamma^+(x_i))+\overline{\vartheta}(\gamma^-(x_i))+\overline{\vartheta}(\gamma^+(x_i))\right)}{4}.$$

Proposition 5. A *RIPJSM* between $\vartheta\left(\widetilde{\Omega_1}\right)$ and $\vartheta\left(\widetilde{\Omega_2}\right)$ satisfies the following properties:

$$0\le \mathrm{D}_{\mathrm{RIPF}}\left(\vartheta\left(\widetilde{\Omega_1}\right),\vartheta\left(\widetilde{\Omega_2}\right)\right)\le 1;$$

$$\mathrm{D}_{\mathrm{RIPF}}\left(\vartheta\left(\widetilde{\Omega_1}\right),\vartheta\left(\widetilde{\Omega_2}\right)\right)=1 \Longleftrightarrow \vartheta\left(\widetilde{\Omega_1}\right)=\vartheta\left(\widetilde{\Omega_2}\right);$$

$$\mathrm{D}_{\mathrm{RIPF}}\left(\vartheta\left(\widetilde{\Omega_1}\right),\vartheta\left(\widetilde{\Omega_2}\right)\right)=\mathrm{D}_{\mathrm{IRPF}}\left(\vartheta\left(\widetilde{\Omega_2}\right),\vartheta\left(\widetilde{\Omega_1}\right)\right).$$

Proof. Proof is similar to *Proposition 3.*

If we consider weight ω_i of each elementx_i, a weighted *RIPDSM* between *RIPF* sets $\vartheta\left(\widetilde{\Omega_1}\right)$ and $\vartheta\left(\widetilde{\Omega_2}\right)$ is defined $\sum_{i=1}^{n}\omega_i=1$.as follows:

$$D_{RIPF}\left(\vartheta(\widetilde{\Omega_1}),\vartheta(\widetilde{\Omega_2})\right)=\frac{1}{n}\sum_{i=1}^{n}\omega_i\frac{2\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)+\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2+(\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i))^2}\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i))^2+(\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i))^2}}. \quad (6)$$

$\omega_i\in[0,1], i=1,2,3\ldots n$ and If we take $\omega_i=\frac{1}{n}, i=1,2,\ldots.n$ then $D_{WRIPF}\left(\vartheta\left(\widetilde{\Omega_1}\right),\vartheta\left(\widetilde{\Omega_2}\right)\right)=D_{RIPF}\left(\vartheta\left(\widetilde{\Omega_2}\right),\vartheta\left(\widetilde{\Omega_1}\right)\right)$.

The weighted *RIPDSM* between two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ also satisfies the following properties.

Proposition 6.

$$0 \leq D_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \leq 1;$$

$$D_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1 \Longleftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2});$$

$$D_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = D_{WRIPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right).$$

6| Decision Making Based on CSM, JSM and DSM under RIPF Environment

This section deals with *RIPSM* between *RIPF* sets to the multi-criteria decision making problem. Assume that $K = \{K_1, K_2, \ldots . K_m\}$ be the set of attributes and $Q = \{Q_1, Q_2, \ldots . Q_n\}$ be the set of alternatives. The proposed decision making approach is described by the following steps.

Algorithm 1. (See *Fig. 1*).

Step 1. Construct the Decision Matrix with RIPF Number. The decision maker forms a decision matrix with respect to n alternatives and m attributes in terms of *RIPF* numbers.

Step 2. Determine RIP Mean Operator.

$$\langle \delta\mu(x_i), \delta\gamma(x_i) \rangle = \left(\frac{\left(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i))\right)}{4}, \frac{\left(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i))\right)}{4} \right)$$

for $i = 1,2, \ldots . n$.

Step 3. Determine the Weights of the Attributes. Assume that the weight of the attributes K_j *(j=1,2,...m)* considered by the decision maker is ω_j *(j=1,2,...m)* where all $\omega_j \in [0,1], j = 1,2,3 \ldots m$ and $\sum_{j=1}^{m} \omega_j = 1$.

Step 4. Determine the Benefit Type Attributes and Cost Type Attributes. Generally, the evaluation attribute can be categorized into two types: benefit type attribute and cost type attribute.

For benefit type attribute: $Z^* = \{max(\mu_{Q_i}), min(\gamma_{Q_i})\}$.

For cost type attribute: $Z^* = \{min(\mu_{Q_i}), max(\gamma_{Q_i})\}$.

Step 5. Determine the Weighted RIPSM of the Alternatives.

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \sum_{i=1}^{n} \omega_i C_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right);$$

$$J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \sum_{i=1}^{n} \omega_i J_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right);$$

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$$D_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \sum_{i=1}^{n} \omega_i D_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right).$$

Step 6. Ranking the Alternatives. The ranking order of all alternatives can be determined based on the descending order of similarity measures.

Step 7. End.

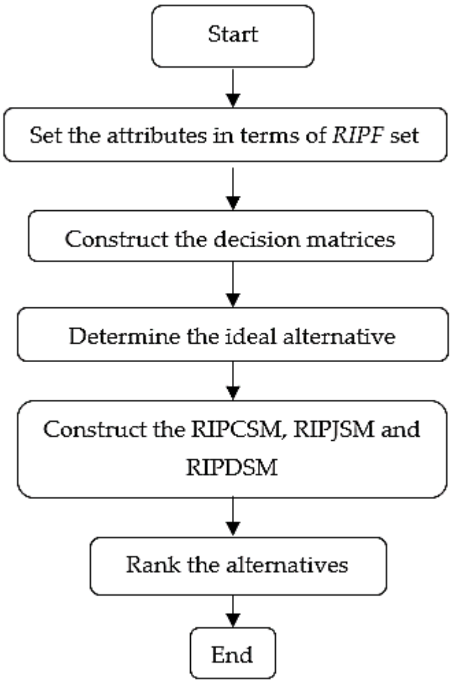

Fig 1. A flowchart of the proposed decision making.

7| Numerical Example for RIPCSM, RIPJSM and RIPDSM

Let us consider a decision maker wants to select the house from $Q = \{Q_1, Q_2, Q_3\}$ by considering four attributes, namely expensive *(K_1)*, reasonable price *(K_2)*, low price *(K_3)* and the risk factor *(K_4)*. By proposed approach discussed above, the considered problem solved by the following steps:

Step 1. The decision maker forms a decision matrix with respect to the three alternatives and four attributes in terms of *RIP* number as follows.

Table 1. Decision matrix.

	K_1	K_2	K_3	K_4
Q_1	([.3,.4],[.5,.7]), ([.3,.4],[.5,.7])	([.5,.6],[.8,.9]), ([.5,.6],[.8,.9])	([.1,.2],[.7,.8]), ([.5,.8],[.4,.6])	([.1,.2],[.7,.8]), ([.5,.8],[.4,.6])
Q_2	([.7,.8],[.6,.7]), ([.7,.8],[.6,.7])	([.7,.8],[.6,.7]), ([.8,.9],[.4,.5])	([.5,.6],[.4,.5]), ([.5,.6],[.4,.5])	([.7,.8],[.6,.7]), ([.8,.9],[.4,.5])
Q_3	([.5,.7],[.3,.4]), ([.8,.9],[.1,.2])	([.5,.7],[.3,.4]), ([.8,.9],[.1,.2])	([.5,.7],[.3,.4]), ([.8,.9],[.1,.2])	([.8,.9],[.1,.2]), ([.8,.9],[.1,.2])

Step 2. Determine the RIP mean operator.

Table 2. Transformed decision matrix.

	K_1	K_2	K_3	K_4
Q_1	[.35,.6]	[.55,.85]	[.4,.625]	[.4,.625]
Q_2	[.75,.65]	[.8,.55]	[.55,.45]	[.8,.55]
Q_3	[.725,.25]	[.725,.25]	[.725,.25]	[.825,.15]

Step 3. The weight vectors considered by the decision maker are 0.35, 0.25, 0.25 and 0.15 respectively.

Step 4. Determine the benefit type attribute and cost type attribute. Here three benefit types attributes K_1,K_2,K_3 and one cost type attribute K_4.

$$Z^* = \{[0.75,0.25],[.8,.25],[.725,.25],[.825,.15]\}.$$

Step 5. Calculate the weighted *RIP* similarity measures of the alternatives. Calculated values of weighted *RIP* similarity values are

$C_{WIRPF}(Q_1, Z^*) = .7582;$

$C_{WIRPF}(Q_2, Z^*) = .9336;$

$C_{WIRPF}(Q_3, Z^*) = .9999;$

$J_{WIRPF}(Q_1, Z^*) = .6046;$

$J_{WIRPF}(Q_2, Z^*) = .8538;$

$J_{WIRPF}(Q_3, Z^*) = .9975;$

$D_{WIRPF}(Q_1, Z^*) = .7018;$

$D_{WIRPF}(Q_2, Z^*) = .9208;$

$D_{WIRPF}(Q_3, Z^*) = .9988.$

Step 6. Ranking the alternatives is prepared based on the descending order of similarity measures. Highest value reflects the best alternative. HencQ_3e is the best alternative.

8| Conclusion

In this paper, we have defined Cosine, Jaccard, Dice similarity measure, Weighted Cosine, Jaccard and Dice similarity measures. We have also proved their basic properties. We have developed MADM strategies based on the proposed measures respectively. We have presented an example for select a best house for live. The thrust of the concept presented in this article will be in pattern recognition, medical diagnosis etc. in rough interval Pythagorean fuzzy sets.

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Spherical Interval-Valued Fuzzy Bi-Ideals of Gamma Near-Rings

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Abstract

In this paper we introduce the concept of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$ and its some results. The union and intersection of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$ is also a spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$. Further we discuss about the relationship between bi-ideal and spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$.

Keywords: Spherical fuzzy set, Interval-Valued fuzzy set, Γ -Near-Rings.

1 | Introduction

The Fuzzy Set (FS) was introduced by Zadeh [14] in 1965. It is identified as better tool for the scientific study of uncertainty, and came as a boost to the researchers working in the field of uncertainty. Many extensions and generalizations of FS was conceived by a number of researchers and a large number of real-life applications were developed in a variety of areas. In addition to this, parallel analysis of the classical results of many branches of Mathematics was also carried out in the fuzzy settings. Properties of fuzzy ideals in near-rings was studied by Hong et al. [3]. The monograph by Chinnadurai [1] gives a detailed discussion on fuzzy ideals in algebraic structures. Fuzzy ideals in Gamma near-ring $\mathscr{R}$ was discussed by Jun et al. [6] and [7] and Satyanarayana [8]. Thillaigovindan et al. [13] studied the interval valued fuzzy quasi-ideals of semigroups. Meenakumari and Tamizh chelvam [9] have defined fuzzy bi-ideal in $\mathscr{R}$ and established some properties of this structure. Srinivas and Nagaiah [11] have proved some results on T-fuzzy ideals of Γ-near-rings.

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Thillaigovindan et al. [12] worked on interval valued fuzzy ideals of near-rings. Chinnadurai and Kadalarasi [2] have defined the direct product of fuzzy ideals in near-rings. Kutlu Gündoğdu and Kahraman [8] introduced spherical fuzzy sets as an extension of picture fuzzy sets. Chinnadurai and Shakila [3] and [4] discussed T-fuzzy bi-ideal of gamma near-ring and spherical fuzzy bi-ideals of gamma near-rings.

In this research work, we introduce the notion of Spherical Interval-Valued Fuzzy Bi-Ideal (SIVFBI) of gamma near-ring $\mathscr{R}$ as a generalization of spherical fuzzy bi-ideals of gamma near-rings $\mathscr{R}$. We will discuss some of the properties of spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathscr{R}$.

2| Preliminaries

In this section we present some definitions which are used for this research. Let $\mathcal{R}$ be a near-ring and Γ be a non-empty set such tha $\mathscr{R}$ t is a Gamma near-ring. A subgroup H of $(\mathscr{R},+)$ is a Bi-Ideal (BI) if and only if $H\Gamma\mathscr{R}\Gamma H \subseteq H$. A Spherical Fuzzy Set (SFS) $\tilde{A}_s$ of the universe of discourse U is given by, $\tilde{A}_s = \{u, (\tilde{\mu}(u), \tilde{\nu}(u), \tilde{\xi}(u)) | u \in U\}$ where $\tilde{\mu}(u): U \longrightarrow [0,1]$, $\tilde{\nu}(u): U \longrightarrow [0,1]$ and $\tilde{\xi}(u): U \longrightarrow [0,1]$ and $0 \leq \tilde{\mu}^2(u) + \tilde{\nu}^2(u) + \tilde{\xi}^2(u) \leq 1$, $u \in U$.

For each u, the numbers $\tilde{\mu}(u)$, $\tilde{\nu}(u)$ and $\tilde{\xi}(u)$ are the degree of membership, non-membership and hesitancy of u to $\tilde{A}_s$, respectively.

A SFS $A_s = (\mu, \nu, \xi)$, where $\mu: \mathscr{R} \longrightarrow [0,1]$, $\nu: \mathscr{R} \longrightarrow [0,1]$ and $\xi: \mathscr{R} \longrightarrow [0,1]$ of $\mathscr{R}$ is said to be a Spherical Fuzzy Bi-Ideal (SFBI) of $\mathscr{R}$ if the following conditions are satisfied

$$\mu(u - v) \geq \min\{\mu(u), \mu(v)\},$$

$$\nu(u - v) \geq \min\{\nu(u), \nu(v)\},$$

$$\xi(u - v) \leq \max\{\xi(u), \xi(v)\},$$

$$\mu(u\alpha v\beta w) \geq \min\{\mu(u), \mu(w)\},$$

$$\nu(u\alpha v\beta w) \geq \min\{\nu(u), \nu(w)\},$$

$$\xi(u\alpha v\beta w) \leq \max\{\xi(u), \xi(w)\},$$

for all $u, v, w \in \mathscr{R}$ and $\alpha, \beta \in \Gamma$.

3| Spherical Interval-Valued Fuzzy Bi-Ideals of Gamma Near-Rings

In this section we define SIVFBI of $\mathscr{R}$ and study some of it properties. We obtain the condition for an arbitrary fuzzy subset of $\mathscr{R}$ is said to be SIVFBI.

Definition 1. A spherical fuzzy *set* $\tilde{A}_s = (\tilde{\mu}, \tilde{\nu}, \tilde{\xi})$ *of* $\mathscr{R}$ is to be SIVFBI of $\mathscr{R}$ if the following conditions are satisfied

$$\tilde{\mu}(u - v) \geq \min^{i}\{\tilde{\mu}(u), \tilde{\mu}(v)\},$$

$$\tilde{\nu}(u - v) \geq \min^{i}\{\tilde{\nu}(u), \tilde{\nu}(v)\},$$

$$\tilde{\xi}(u - v) \leq \max^{i}\{\tilde{\xi}(u), \tilde{\xi}(v)\},$$

$$\tilde{\mu}(u\alpha v\beta w) \geq \min^{i}\{\tilde{\mu}(u), \tilde{\mu}(w)\},$$

$$\tilde{v}(u\alpha v\beta w) \geq \min^{i}\{\tilde{v}(u), \tilde{v}(w)\},$$

$$\tilde{\xi}(u\alpha v\beta w) \leq \max^{i}\{\tilde{\xi}(u), \tilde{\xi}(w)\},$$

for all $u, v, w \in \mathscr{R}$ and $\alpha, \beta \in \Gamma$, where $\tilde{\mu}: \mathscr{R} \longrightarrow D[0,1]$, $\tilde{v}: \mathscr{R} \longrightarrow D[0,1]$ and $\tilde{\xi}: \mathscr{R} \longrightarrow D[0,1]$. Here $D[0,1]$ denotes the family of closed subintervals of [0,1].

Example 1. Let $\mathscr{R} = \{0,1,2,3\}$ with binary operation $+$" on $\mathscr{R}, \Gamma = \{0,1\}$ and $\mathscr{R} \times \Gamma \times \mathscr{R} \longrightarrow \mathscr{R}$ be a mapping. From the cayley table,

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1

0	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	2	2	2
3	0	3	3	3

1	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Define SFS $\tilde{\mu}: \mathscr{R} \longrightarrow D[0,1]$ by $\tilde{\mu}(0) = [0.2,0.3], \tilde{\mu}(1) = [0.3,0.6], \tilde{\mu}(2) = [0.7,0.9], \tilde{\mu}(3) = [0.5,0.9]$; $\tilde{v}: \mathscr{R} \longrightarrow D[0,1]$ by $\tilde{v}(0) = [0.2,0.4], \tilde{v}(1) = [0.5,0.6], \tilde{v}(2) = [0.6,0.7], \tilde{v}(3) = [0.7,0.9]$; $\tilde{\xi}: \mathscr{R} \longrightarrow D[0,1]$ by $\tilde{\xi}(0) = [0.1,0.3], \tilde{\xi}(1) = [0.4,0.6], \tilde{\xi}(2) = [0.8,0.9], \tilde{\xi}(3) = [0.5,0.7]$. Then $\tilde{A}_s$ is SIVFBI of $\mathscr{R}$.

Theorem 1. Let $\tilde{A}_s = [A_s^-; A_s^+]$ be a Spherical interval-valued fuzzy subset of a gamma near-ring $\mathscr{R}$, then $\tilde{A}_s$ is a SIVFBI of $\mathscr{R}$ if and only if A_s^-, A_s^+ are SFBI of $\mathscr{R}$.

Proof. If $\tilde{A}_s$ is a SIVFBI of $\mathscr{R}$. For any $u, v, w \in \mathscr{R}$. Now,

$$[\mu^-(u-v), \mu^+(u-v)] = \tilde{\mu}(u-v)$$

$$\geq \min^{i}\{\tilde{\mu}(u), \tilde{\mu}(v)\}$$

$$= \min^{i}\{[\mu^-(u), \mu^+(u)], [\mu^-(v), \mu^+(v)]\}$$

$$= \min^{i}\{[\mu^-(u), \mu^-(v)]\}, \min^{i}\{[\mu^+(u), \mu^+(v)]\},$$

$$[v^-(u-v), v^+(u-v)] = \tilde{v}(x-y)$$

$$\geq \min^{i}\{\tilde{v}(u), \tilde{v}(v)\}$$

$$= \min^{i}\{[v^-(u), v^+(u)], [v^-(v), v^+(v)]\}$$

$$= \min^{i}\{[v^-(u), v^-(v)]\}, \min^{i}\{[v^+(u), v^+(v)]\}, \text{ and}$$

$$[\xi^-(u-v), \xi^+(u-v)] = \tilde{\xi}(u-v)$$

$\leq \max^{i}\{\tilde{\xi}(u), \tilde{\xi}(v)\}$

$= \max^{i}\{[\xi^{-}(u), \xi^{+}(u)], [\xi^{-}(v), \xi^{+}(v)]\}$

$= \max^{i}\{[\xi^{-}(u), \xi^{-}(v)]\}, \max^{i}\{[\xi^{+}(u), \xi^{+}(v)]\};$

$[\mu^{-}(u\alpha v\beta w), \mu^{+}(u\alpha v\beta w)] = \tilde{\mu}(u\alpha v\beta w)$

$\geq \min^{i}\{\tilde{\mu}(u), \tilde{\mu}(w)\}$

$= \min^{i}\{[\mu^{-}(u), \mu^{+}(u)], [\mu^{-}(w), \mu^{+}(w)]\}$

$= \min^{i}\{[\mu^{-}(u), \mu^{-}(w)]\}, \min^{i}\{[\mu^{+}(u), \mu^{+}(w)]\};$

$[\nu^{-}(u\alpha v\beta w), \nu^{+}(u\alpha v\beta w)] = \tilde{\nu}(u\alpha v\beta w)$

$\geq \min^{i}\{\tilde{\nu}(u), \tilde{\nu}(w)\}$

$= \min^{i}\{[\nu^{-}(u), \nu^{+}(u)], [\nu^{-}(w), \nu^{+}(w)]\}$

$= \min^{i}\{[\nu^{-}(u), \nu^{-}(w)]\}, \min^{i}\{[\nu^{+}(u), \nu^{+}(w)]\}$, and

$[\xi^{-}(u\alpha v\beta w), \xi^{+}(u\alpha v\beta w)] = \tilde{\xi}(u\alpha v\beta w)$

$\leq \max^{i}\{\tilde{\xi}(u), \tilde{\xi}(w)\}$

$= \max^{i}\{[\xi^{-}(u), \xi^{+}(u)], [\xi^{-}(w), \xi^{+}(w)]\}$

$= \max^{i}\{[\xi^{-}(u), \xi^{-}(w)]\}, \max^{i}\{[\xi^{+}(u), \xi^{+}(w)]\}.$

Therefore A_s^-, A_s^+ are SFBI o $\mathscr{R}$ f.

Conversely let A_s^-, A_s^+ are SFBI of $\mathscr{R}$. Let $u, v, w \in \mathscr{R}$. Now,

$\tilde{\mu}(u - v) = [\mu^{-}(u - v), \mu^{+}(u - v)]$

$\geq [\min^{i}\{\mu^{-}(u), \mu^{-}(v)\}, \min^{i}\{\mu^{+}(u), \mu^{+}(v)\}]$

$= \min^{i}\{\mu^{-}(u), \mu^{+}(u)\}, \min^{i}\{\mu^{-}(v), \mu^{+}(v)\}$

$= \min^{i}\{\tilde{\mu}(u), \tilde{\mu}(v)\};$

$\tilde{\nu}(u - v) = [\nu^{-}(u - v), \nu^{+}(u - v)]$

$\geq [\min^{i}\{\nu^{-}(u), \nu^{-}(v)\}, \min^{i}\{\nu^{+}(u), \nu^{+}(v)\}]$

$= \min^{i}\{\nu^{-}(u), \nu^{+}(u)\}, \min^{i}\{\nu^{-}(v), \nu^{+}(v)\}$

$= \min^{i}\{\tilde{\nu}(u), \tilde{\nu}(v)\}$, and

$\tilde{\xi}(u - v) = [\xi^{-}(u - v), \xi^{+}(u - v)]$

$\leq [\max^i\{\xi^-(u),\xi^-(v)\},\max^i\{\xi^+(u),\xi^+(v)\}]$

$= \max^i\{\xi^-(u),\xi^+(u)\},\max^i\{\xi^-(v),\xi^+(v)\}$

$= \max^i\{\tilde{\xi}(u),\tilde{\xi}(v)\};$

$\tilde{\mu}(u\alpha v\beta w) = [\mu^-(u\alpha v\beta w),\mu^+(u\alpha v\beta w)]$

$\geq [\min^i\{\mu^-(u),\mu^-(w)\},\min^i\{\mu^+(u),\mu^+(w)\}]$

$= \min^i\{\mu^-(u),\mu^+(u)\},\min^i\{\mu^-(w),\mu^+(w)\}$

$= \min^i\{\tilde{\mu}(u),\tilde{\mu}(w)\};$

$\tilde{\nu}(u\alpha v\beta w) = [\nu^-(u\alpha v\beta w),\nu^+(u\alpha v\beta w)]$

$\geq [\min^i\{\nu^-(u),\nu^-(w)\},\min^i\{\nu^+(u),\nu^+(w)\}]$

$= \min^i\{\nu^-(u),\nu^+(u)\},\min^i\{\nu^-(w),\nu^+(w)\}$

$= \min^i\{\tilde{\nu}(u),\tilde{\nu}(w)\}$, and

$\tilde{\xi}(u\alpha v\beta w) = [\xi^-(u\alpha v\beta w),\xi^+(u\alpha v\beta w)]$

$\leq [\max^i\{\xi^-(u),\xi^-(w)\},\max^i\{\xi^+(u),\xi^+(w)\}]$

$= \max^i\{\xi^-(u),\xi^+(u)\},\max^i\{\xi^-(w),\xi^+(w)\}$

$= \max^i\{\tilde{\xi}(u),\tilde{\xi}(w)\}.$

So $\tilde{A}_s$ is a SIVFBI of $\mathscr{R}$.

Hence the proof.

Theorem 2. If $\{\tilde{A}_{s_i}; i \in I\}$ be a family of SIVFBI of a gamma near-ring $\mathscr{R}$, then $\bigcap_{i\in I} \tilde{A}_{s_i}$ is also SIVFBI of $\mathscr{R}$, where I is an index set.

Proof. Let $\{\tilde{A}_{s_i}; i \in I\}$ be a family of SIVFBI of a gamma near-ring $\mathscr{R}$. For any $u, v, w \in \mathscr{R}$ and $\alpha, \beta \in \Gamma$.

$\bigcap_{i\in I} \tilde{\mu}_i(u-v) = \inf^i_{i\in I}\tilde{\mu}_i(u-v)$

$\geq \inf^i_{i\in I}\min^i\{\tilde{\mu}_i(u),\tilde{\mu}_i(v)\}$

$= \min^i\{\inf^i_{i\in I}\tilde{\mu}_i(u),\inf^i_{i\in I}\tilde{\mu}_i(v)\}$

$= \min^i\{\bigcap_{i\in I} \tilde{\mu}_i(u),\bigcap_{i\in I} \tilde{\mu}_i(v)\};$

$\bigcap_{i\in I} \tilde{\nu}_i(u-v) = \inf^i_{i\in I}\tilde{\nu}_i(u-v)$

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$$\geq \inf^{i}_{i\in I}\min^{i}\{\tilde{\nu}_i(u),\tilde{\nu}_i(v)\}$$

$$= \min^{i}\{\inf^{i}_{i\in I}\tilde{\nu}_i(u),\inf^{i}_{i\in I}\tilde{\nu}_i(v)\}$$

$$= \min^{i}\{\bigcap_{i\in I}\tilde{\nu}_i(u),\bigcap_{i\in I}\tilde{\nu}_i(v)\};$$

$$\bigcap_{i\in I}\tilde{\xi}_i(u-v) = \inf^{i}_{i\in I}\tilde{\xi}_i(u-v)$$

$$\leq \inf^{i}_{i\in I}\max^{i}\{\tilde{\xi}_i(u),\tilde{\xi}_i(v)\}$$

$$= \max^{i}\{\inf^{i}_{i\in I}\tilde{\xi}_i(u),\inf^{i}_{i\in I}\tilde{\xi}_i(v)\}$$

$$= \max^{i}\{\bigcap_{i\in I}\tilde{\xi}_i(u),\bigcap_{i\in I}\tilde{\xi}_i(v)\};$$

$$\bigcap_{i\in I}\tilde{\mu}_i(u\alpha v\beta w) = \inf^{i}_{i\in I}\tilde{\mu}_i(u\alpha v\beta w)$$

$$\geq \inf^{i}_{i\in I}\min^{i}\{\tilde{\mu}_i(u),\tilde{\mu}_i(w)\}$$

$$= \min^{i}\{\inf^{i}_{i\in I}\tilde{\mu}_i(u),\inf^{i}_{i\in I}\tilde{\mu}_i(w)\}$$

$$= \min^{i}\{\bigcap_{i\in I}\tilde{\mu}_i(u),\bigcap_{i\in I}\tilde{\mu}_i(w)\};$$

$$\bigcap_{i\in I}\tilde{\nu}_i(u\alpha v\beta w) = \inf^{i}_{i\in I}\tilde{\nu}_i(u\alpha v\beta w)$$

$$\geq \inf^{i}_{i\in I}\min^{i}\{\tilde{\nu}_i(u),\tilde{\nu}_i(w)\}$$

$$= \min^{i}\{\inf^{i}_{i\in I}\tilde{\nu}_i(u),\inf^{i}_{i\in I}\tilde{\nu}_i(w)\}$$

$$= \min^{i}\{\bigcap_{i\in I}\tilde{\nu}_i(u),\bigcap_{i\in I}\tilde{\nu}_i(w)\}, \text{ and}$$

$$\bigcap_{i\in I}\tilde{\xi}_i(u\alpha v\beta w) = \inf^{i}_{i\in I}\tilde{\xi}_i(u\alpha v\beta w)$$

$$\leq \inf^{i}_{i\in I}\max^{i}\{\tilde{\xi}_i(u),\tilde{\xi}_i(w)\}$$

$$= \max^{i}\{\inf^{i}_{i\in I}\tilde{\xi}_i(u),\inf^{i}_{i\in I}\tilde{\xi}_i(w)\}$$

$$= \max^{i}\{\bigcap_{i\in I}\tilde{\xi}_i(u),\bigcap_{i\in I}\tilde{\xi}_i(w)\}.$$

Hence the proof.

Theorem 3. If $\{\tilde{A}_{s_i}; i \in I\}$ be a family of SIVFBI of a gamma near-ring $\mathscr{R}$, then $\bigcup_{i\in I}\tilde{A}_{s_i}$ is also SIVFBI of $\mathscr{R}$, where I is an index set.

Proof. Let $\{\tilde{A}_{s_i}; i \in I\}$ be a family of SIVFBI of a gamma near-ring $\mathscr{R}$. For any $u, v, w \in \mathscr{R}$ and $\alpha, \beta \in \Gamma$.

$$\bigcup_{i\in I}\tilde{\mu}_i(u-v)=\sup^i_{i\in I}\tilde{\mu}_i(u-v)$$

$$\geq \sup^i_{i\in I}\min^i\{\tilde{\mu}_i(u),\tilde{\mu}_i(v)\}$$

$$=\min^i\{\sup^i_{i\in I}\tilde{\mu}_i(u),\sup^i_{i\in I}\tilde{\mu}_i(v)\}$$

$$=\min^i\{\bigcup_{i\in I}\tilde{\mu}_i(u),\bigcup_{i\in I}\tilde{\mu}_i(v)\};$$

$$\bigcup_{i\in I}\tilde{\nu}_i(u-v)=\sup^i_{i\in I}\tilde{\nu}_i(u-v)$$

$$\geq \sup^i_{i\in I}\min^i\{\tilde{\nu}_i(u),\tilde{\nu}_i(v)\}$$

$$=\min^i\{\sup^i_{i\in I}\tilde{\nu}_i(u),\sup^i_{i\in I}\tilde{\nu}_i(v)\}$$

$$=\min^i\{\bigcup_{i\in I}\tilde{\nu}_i(u),\bigcup_{i\in I}\tilde{\nu}_i(v)\};$$

$$\bigcup_{i\in I}\tilde{\xi}_i(u-v)=\inf^i_{i\in I}\tilde{\xi}_i(u-v)$$

$$\leq \sup^i_{i\in I}\max^i\{\tilde{\xi}_i(u),\tilde{\xi}_i(v)\}$$

$$=\max^i\{\sup^i_{i\in I}\tilde{\xi}_i(u),\sup^i_{i\in I}\tilde{\xi}_i(v)\}$$

$$=\max^i\{\bigcup_{i\in I}\tilde{\xi}_i(u),\bigcup_{i\in I}\tilde{\xi}_i(v)\};$$

$$\bigcup_{i\in I}\tilde{\mu}_i(u\alpha v\beta w)=\sup^i_{i\in I}\tilde{\mu}_i(u\alpha v\beta w)$$

$$\geq \sup^i_{i\in I}\min^i\{\tilde{\mu}_i(u),\tilde{\mu}_i(w)\}$$

$$=\min^i\{\sup^i_{i\in I}\tilde{\mu}_i(u),\sup^i_{i\in I}\tilde{\mu}_i(w)\}$$

$$=\min^i\{\bigcap_{i\in I}\tilde{\mu}_i(u),\bigcup_{i\in I}\tilde{\mu}_i(w)\};$$

$$\bigcup_{i\in I}\tilde{\nu}_i(u\alpha v\beta w)=\sup^i_{i\in I}\tilde{\nu}_i(u\alpha v\beta w)$$

$$\geq \sup^i_{i\in I}\min^i\{\tilde{\nu}_i(u),\tilde{\nu}_i(w)\}$$

$$=\min^i\{\sup^i_{i\in I}\tilde{\nu}_i(u),\sup^i_{i\in I}\tilde{\nu}_i(w)\}$$

$$=\min^i\{\bigcup_{i\in I}\tilde{\nu}_i(u),\bigcup_{i\in I}\tilde{\nu}_i(w)\},\text{ and}$$

$$\bigcup_{i\in I}\tilde{\xi}_i(u\alpha v\beta w)=\sup^i_{i\in I}\tilde{\xi}_i(u\alpha v\beta w)$$

$$\leq \inf^i_{i\in I}\max^i\{\tilde{\xi}_i(u),\tilde{\xi}_i(w)\}$$

$$=\max^i\{\sup^i_{i\in I}\tilde{\xi}_i(u),\sup^i_{i\in I}\tilde{\xi}_i(w)\}$$

$$= \max^{i}\{\bigcup_{i\in I} \tilde{\xi}_i(u), \bigcup_{i\in I} \tilde{\xi}_i(w)\}.$$

Hence the proof.

Theorem 4. If $\tilde{A}_s$ and $\tilde{\sigma}_s$ are SIVFBIs of $\mathscr{R}$, then $\tilde{A}_s \wedge \tilde{\sigma}_s$ is SIVFBI of $\mathscr{R}$.

Proof. Let and $\tilde{\sigma}_s$ are SFBIs of $\mathscr{R}$. Let $u, v, w \in \mathscr{R}$ and $\alpha, \beta \in \Gamma$. Then,

$(\tilde{\mu} \wedge \tilde{\sigma}_s)(u - v) = min^i\{\tilde{\mu}(u - v), \tilde{\sigma}_s(u - v)\}$, since by $(\tilde{\mu} \wedge \tilde{\sigma})(u) = min^i\{\tilde{\mu}(u), \tilde{\sigma}(u)\}$

$$\geq \min^i\{\min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}, \min^i\{\tilde{\sigma}_s(u), \tilde{\sigma}_s(v)\}\}$$

$$= \min^i\{\min^i\{\min^i\{\tilde{\mu}(u), \tilde{\mu}(v), \tilde{\sigma}_s(u)\}, \tilde{\sigma}_s(v)\}\}$$

$$= \min^i\{\min^i\{\min^i\{\tilde{\mu}(u), \tilde{\sigma}_s(u)\}, \tilde{\mu}(v)\}, \tilde{\sigma}_s(v)\}$$

$$= \min^i\{\min^i\{\tilde{\mu}(u), \tilde{\sigma}_s(u)\}, \min^i\{\tilde{\mu}(v), \tilde{\sigma}_s(v)\}$$

$$= \min^i\{\tilde{\mu} \wedge \tilde{\sigma}(u)), (\tilde{\mu} \wedge \tilde{\sigma}(v))\}.$$

Also $(\tilde{\nu} \wedge \tilde{\sigma}_s)(u - v) \geq \min^i\{\tilde{\nu} \wedge \tilde{\sigma}(u)), (\tilde{\nu} \wedge \tilde{\sigma}(v))\}$ and $(\tilde{\xi} \wedge \tilde{\sigma}_s)(u - v) \leq \max^i\{\tilde{\xi} \wedge \tilde{\sigma}(u)), (\tilde{\xi} \wedge \tilde{\sigma}(v))\}$.

Since $(\tilde{\mu}(u\alpha v\beta w) \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v))$.

$$(\tilde{\mu} \wedge \tilde{\sigma}_s)(u\alpha v\beta w) = \min^i\{\tilde{\mu}(u\alpha v\beta w), \tilde{\sigma}_s(u\alpha v\beta w))$$

$$\geq \min^i\{\min^i\{\tilde{\mu}(u), \widetilde{A}_s(w)\}, \min^i\{\tilde{\sigma}_s(u), \tilde{\sigma}_s(w)\}\}$$

$$= \min^i\{\min^i\{\tilde{\mu}(u), \tilde{\sigma}_s(u)\}, \min^i\{\tilde{\mu}(w)\}, \tilde{\sigma}_s(w)\}\}$$

$$= \min^i\{(\tilde{\mu} \wedge \tilde{\sigma}_s)(u), (\tilde{\mu} \wedge \tilde{\sigma}_s)(w)\}.$$

Also $(\tilde{\nu} \wedge \tilde{\sigma}_s)(u\alpha v\beta w) \geq min^i\{(\tilde{\nu} \wedge \tilde{\sigma}_s)(u), (\tilde{\nu} \wedge \tilde{\sigma}_s)(w)$ and $(\tilde{\xi} \wedge \tilde{\sigma}_s)(u\alpha v\beta w) \leq max^i\{(\tilde{\xi} \wedge \tilde{\sigma}_s)(u), (\tilde{\xi} \wedge \tilde{\sigma}_s)(w)\}$.

Hence $(\tilde{A}_s \wedge \tilde{\sigma}_s)$ is a SIVFBI of $\mathscr{R}$.

Lemma 1. Let A be BI of $\mathscr{R}$. For any $0 < m < 1$, there exists a SIVFBI A_s of $\mathscr{R}$ such that $\tilde{A}_{s_m} = A$.

Proof. Let A be BI of $\mathscr{R}$. Define $\tilde{A}_s: \mathscr{R} \longrightarrow [0,1]$ by

$$\widetilde{A}_s(u) = \begin{cases} m, & \text{if } u \in A \\ 0, & \text{if } u \notin A. \end{cases}$$

where m be a constant in (0,1). Clearly $\widetilde{A}_{s_m} = A$. Let $u, v \in \mathscr{R}$. If $u, v \in A$, then $\tilde{\mu}(u - v) = m \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}, \widetilde{\nu}(u - v) = m \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(v)\}$ and $\tilde{\xi}(u - v) = m \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\}$.

If at least one of u and v is not in A, then $u - v \notin A$ and so $\tilde{\mu}(u - v) = 0 = min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}$, $\tilde{\nu}(u - v) = 0 = min^i\{\tilde{\nu}(u), \tilde{\nu}(v)\}$ and $\tilde{\xi}(u - v) = 0 = max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\}$.

Let $u, v, w \in \mathscr{R}$ and $\alpha, \beta \in \Gamma$. If $u, w \in A$, then $\tilde{\mu}(u), \tilde{\nu}(u), \tilde{\xi}(u) = m; \tilde{\mu}(w), \tilde{\nu}(w), \tilde{\xi}(w) = m$. Also $\tilde{\mu}(u\alpha v\beta w) = m \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$, $\tilde{\nu}(u\alpha v\beta w) = m \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\}$ and $\tilde{\xi}(u\alpha v\beta w) = m \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\}$.

If at least one of u and w is not in A, then $\tilde{\mu}(u\alpha v\beta w) \geq 0 = min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$, $\tilde{\nu}(u\alpha v\beta w) \geq 0 = min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\}$ and $\tilde{\xi}(u\alpha v\beta w) \leq 0 = max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\}$.

Thus $\widetilde{A}_s$ is SIVFBI of $\mathscr{R}$.

Theorem 5. If $\tilde{A}_s$ be SIVFBI of $\mathscr{R}$, then the complement $\tilde{A}_s$ is also SIVFBI o $\mathscr{R}$ f.

Proof. For $u, v, w \in \mathscr{R}$ and $\alpha, \beta \in \Gamma$, we have

$\tilde{\mu}(u-v) = 1 - \tilde{\mu}(u-v) \geq 1 - min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}$=$min^i\{1-\tilde{\mu}(u), 1-\tilde{\mu}(v)\}$= $min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}$, and also $\tilde{\nu}(u-v) \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(v)\}$, $\tilde{\xi}(u-v) \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\}$.

$\tilde{\mu}(u\alpha v\beta w) = 1 - \tilde{\mu}(u\alpha v\beta w) \geq 1 - min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$=$min^i\{1-\tilde{\mu}(u), 1-\tilde{\mu}(w)\}$= $min^i\{\tilde{\mu}(u), \tilde{\mu}(w))$, and also $\tilde{\nu}(u\alpha v\beta w) \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\}$, $\tilde{\xi}(u\alpha v\beta w) \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\}$.

Hence $\tilde{A}_s$ is also SIVFBI of $\mathscr{R}$.

Lemma 6. Let U is fuzzy subset of $\mathscr{R}$. Then U is BI of $\mathscr{R}$ if and only if $\tilde{A}_{s_U}$ is SIVFBI of $\mathscr{R}$.

Proof. Let U be BI of $\mathscr{R}$. For $u, v \in U$, $u - v \in U$.

Let $u, v \in \mathscr{R}$.

case(a): If $u, v \in U$, then $\tilde{\mu}_U(u) = 1$ and $\tilde{\mu}_U(v) = 1$. Thus $\tilde{\mu}_U(u-v) = 1 \geq min^i\left\{\tilde{\mu}(u), \tilde{\mu}(v)\right\}$.

case(b): If $u \in U$ and $v \notin U$, then $\tilde{\mu}_U(u) = 1$ and $\tilde{\mu}_U(v) = 0$. Thus $\tilde{\mu}_U(u-v) = 0 \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v))$.

case(c): If $u \notin U$ and $v \in U$, then $\tilde{\mu}_U(u) = 0$ and $\tilde{\mu}_U(v) = 1$. Thus $\tilde{\mu}_U(u-v) = 0 \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}$.

case(d): If $u \notin U$ and $v \notin U$, then $\tilde{\mu}_U(u) = 0$ and $\tilde{\mu}_U(v\tilde{\xi}_U(u-v) \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\}) = 0$. Thu$\widetilde{\mu}_U(u-v) = 0s \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}$.

In the above four cases $\tilde{\nu}_U(u-v) \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(v)\}$.

Let $u, v, w \in \mathscr{R}$.

case(a): If $u \in U$ and $w \in U$, then $\tilde{\mu}_U(u) = 1$ and $\tilde{\mu}_U(w) = 1$. Thus $\tilde{\mu}_U(u\alpha v\beta w) = 1 \geq min^i\{\tilde{A}_{s_U}\tilde{\mu}(u), \tilde{\mu}(w)\}$.

case(b): If $u \in U$ an $w \notin U$ d, then $\widetilde{\mu}_U(u) = 1$ and $\tilde{\mu}_U(w) = 0$. Thus $\tilde{\mu}_U(u\alpha v\beta w) = 0 \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$.

case(c): If $u \notin U$ and $w \in U$, then $\tilde{\mu}_U(u) = 0$ and $\tilde{\mu}_U(w) = 1$. Thus $\mu_U(u\alpha v\beta w) = 0 \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$.

case(d): If $u \notin U$ and $w \notin U$, then $\tilde{\mu}_U(u) = 0$ and $\tilde{\mu}_U(w) = 0$. Thus $\tilde{\mu}_U(u\alpha v\beta w) = 0 \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$.

Also $\tilde{\nu}_U(u\alpha v\beta w) \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\}$ and$\tilde{\xi}_U(u\alpha v\beta w) \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\}$ Thus $\tilde{A}_{s_U}$ is a SIVFBI of $\mathscr{R}$.

Conversely, suppose is a SIVFBI of $\mathscr{R}$. Then by *Lemma 4* $\tilde{A}_{s_U}$ has only two elements.

Hence U is BI of $\mathscr{R}$.

Theorem 7. If $\mathscr{R}$ be a gamma near-ring and $\tilde{A}_s$ be SIVFBI of $\mathscr{R}$, then the set $\mathscr{R}_{\tilde{A}_s} = \{u \in \mathscr{R} | \tilde{A}_s(u) = \tilde{A}_s(0)\}$ is BI of $\mathscr{R}$.

Proof. Let $\tilde{A}_s$ be SIVFBI of and le $u, v, w \in \mathscr{R}$ t. Then

$\tilde{\mu}(u-v) \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\} = min^i\{\tilde{\mu}(0), \tilde{\mu}(0)\} = \tilde{\mu}(0)$. So $\tilde{\mu}(u-v) = \tilde{\mu}(0)$, *then* $u - v \in \mathscr{R}_{\tilde{A}_s}$.

$\tilde{v}(u-v) \geq min^i\{\tilde{v}(u), \tilde{v}(v)\} = min^i\{\tilde{v}(0), \tilde{v}(0)\} = \tilde{v}(0)$. So $\tilde{v}(u-v) = \tilde{v}(0)$, *then* $u - v \in \mathscr{R}_{\tilde{A}_s}$.

$\tilde{\xi}(u-v) \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\} = max^i\{\tilde{\xi}(0), \tilde{\xi}(0)\} = \tilde{\xi}(0)$. So $\tilde{\xi}(u-v) = \tilde{\xi}(0)$, *then* $u - v \in \mathscr{R}_{\tilde{A}_s}$.

$\tilde{\mu}(u\alpha v\beta w) \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\} = min^i\{\tilde{\mu}(0), \tilde{\mu}(0)\} = \tilde{\mu}(0)$. So $\tilde{\mu}(u\alpha v\beta w) = \tilde{\mu}(0)$, *then* $u\alpha v\beta w \in \mathscr{R}_{\tilde{A}_s}$.

$\tilde{v}(u\alpha v\beta w) \geq min^i\{\tilde{v}(u), \tilde{v}(w)\} = min^i\{\tilde{v}(0), \tilde{v}(0)\} = \tilde{v}(0)$. So $\tilde{v}(u\alpha v\beta w) = \tilde{v}(0)$, *then* $u\alpha v\beta w \in \mathscr{R}_{\tilde{A}_s}$.

$\tilde{\xi}(u\alpha v\beta w) \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\} = max^i\{\tilde{\xi}(0), \tilde{\xi}(0)\} = \tilde{\xi}(0)$. So $\tilde{\xi}(u\alpha v\beta w) = \tilde{\xi}(0)$, *then* $u\alpha v\beta w \in \mathscr{R}_{\tilde{A}_s}$.

Then $\mathscr{R}_{\tilde{A}_s}$ is BI of $\mathscr{R}$.

Theorem 8. If B be a non-empty subset of $\mathscr{R}$ and $\widetilde{A}_{s_B}$ be a Spherical Interval-Valued Fuzzy Set (SIVFS) $\mathscr{R}$ defined by

$$\widetilde{A}_{s_B}(u) = \begin{cases} \tilde{p}, & \text{if } u \in B \\ \tilde{q}, & \text{otherw}\tilde{p} \geq \tilde{q}\text{ise.} \end{cases}$$

for $u \in \mathscr{R}$, $\tilde{p}, \tilde{q} \in D[0,1]$ and. Then $\tilde{A}_{s_B}(u)$ is a SIVFBI of $\mathscr{R}$ if and only if B is a BI of $\mathscr{R}$. Als$\mathscr{R}_{\tilde{A}_{s_B}} = B$ *o.*

Proof. Let $\widetilde{A}_{s_B}$t be a SIVFS $\mathcal{R}$ and le $u, v, w \in B$ t. Then $\tilde{A}_{s_B}(u) = \tilde{p} = \tilde{A}_{s_B}(v) = \tilde{A}_{s_B}(w)$. Now,

$$\widetilde{A}_{s_B}(u-v) \geq \min^i\{\widetilde{A}_{s_B}(u), \widetilde{A}_{s_B}(v)\}$$

$$= \min^i\{\tilde{p}, \tilde{p}\} = \tilde{p}.$$

$$\widetilde{A}_{s_B}(u-v) = \tilde{p}, \text{so } u - v \in B.$$

$$\widetilde{A}_{s_B}(u\alpha v\beta w) \geq \min^i\{\widetilde{A}_{s_B}(u), \widetilde{A}_{s_B}(w)\}$$

$$= \min^i\{\tilde{p}, \tilde{p}\} = \tilde{p}.$$

$$\widetilde{A}_{s_B}B(u\alpha v\beta w) = \tilde{p}, \text{so } u\alpha v\beta w \in B.$$

Then is a BI of $\mathscr{R}$.

Conversely let B be a BI of $\mathscr{R}$ and le $u, v, w \in \mathscr{R}$ t.

If at $\tilde{A}_{s_B}(u)$ least one u, v is not in B, then $u - v \notin B$ and so$\tilde{A}_{s_B}(u-v) \geq min^i\{\tilde{A}_{s_B}(u), \tilde{A}_{s_B}(v)\} = \tilde{q}$.

If at least one u, w is not in B, then $u\alpha v\beta w \notin B$ and so $\tilde{A}_{s_B}(u\alpha v\beta w) \geq min^i\{\tilde{A}_{s_B}(u), \tilde{A}_{s_B}(w)\} = \tilde{q}$.

Thus is a SIVFBI of $\mathscr{R}$.

4| Conclusion

We obtained the union and intersection of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathscr{R}$ is also a spherical interval-valued fuzzy bi-ideal of gamma near-ring. And for that condition, $\tilde{A}_{s_m} = A$, for any $0 < \mathrm{m} < 1$, bi-ideal of gamma near-ring $\mathscr{R}$ becomes spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathscr{R}$. In future we will discuss the spherical fuzzy sets in some other algebraic structures.

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K-algebras on Quadripartitioned Single Valued Neutrosophic Sets

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Abstract

Quadripartitioned Single Valued Neutrosophic (QSVN) set is a powerful structure where we have four components: Truth-T, Falsity-F, Unknown-U and Contradiction-C. And also it generalizes the concept of fuzzy, initutionstic and single valued neutrosophic set. In this paper we have proposed the concept of K-algebras on QSVN, level subset of QSVN and studied some of the results. In addition to this we have also investigated the characteristics of QSVN K-subalgebras under homomorphism.

Keywords: Quadripartitioned Single Valued Neutrosophic Set (QSVNS), K-Algebras, Homomorphism, Quadripartitioned single valued neutrosophic K-Algebras.

1 | Introduction

Dar and Akram [12] proposed a novel logical algebra known as K-algebra. The algebraic structure of a group G which K-algebra was built on should have a right identity element and satisfy the properties of non-commutative and non-associative. Furthermore this group G is of the type where each non-identity element is not of order 2 and K-algebra was built by adjoining the induced binary operation on G [11]-[13]. Zadeh's fuzzy set theory [22] was a powerful framework which deals the concept of uncertainty, imprecision and also it represented by membership function which lies in a unit interval of [0,1]. Fuzzy K-algebra was introduced by Akram et al. [2], [3], [5] and also they established this in a wide-reaching way through other researchers. Later Atanassov [9] introduced the concept of intuitionistic fuzzy set in 1983. It has an additional degree called the degree of nonmembership. Intuitionistic fuzzy K-subalgebras was proposed by Akram et al. [4] and [6]. Intuitionistic fuzzy Ideals of BCK-Algebras was proposed by Jun and Kim [14].

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Neutrosophic set which is a generalization of fuzzy set and intuitionistic fuzzy set was introduced by Smarandache [20] in 1998. Along with membership and non-membership function neutrosophic set has one more extra component called indeterminacy membership function. Also all the values of these three components lie in the real standard or non-standard subset of unit interval $]-0,1+[$ where $-0 = 0 - \epsilon$, $1+ = 1 + \epsilon$, ϵ is an infinitesimal number. In neutrosophic set theory algebraic structures were studied in soft topological K-algebras [7]. Agboola and Davvaz [1] presented the introduction to neutrosophic BCI/BCK algebras. Smarandache and Wang et al. [21] introduced single-valued neutrosophic set which plays a vital place in many real life problems and it takes the values from the subset of [0,1]. Akram et al. [8] studied K-algebras on single valued neutrosophic sets and also discussed homomorphisms between the single valued neutrosophic K-subalgebras. Belnap [10] introduced the concept of four valued logic that is the information are represented by four components T, F, None, Both which denote true, false, neither true nor false, both true and false, respectively. Based on this concept, Smarandache proposed four numerical valued neutrosophic logic where indeterminacy is splitted into two terms known as Contradiction (C) and Unknown (U). Chatterjee et al. [19] introduced Quadripartitioned Single Valued Neutrosophic (QSVN) set in which we have four components T, C, U, and F, respectively, and also it lies in the real unit interval of [0,1]. K. Mohana and M. Mohanasundari [15] and [17] studied the concept of Quadripartitioned Single Valued Neutrosophic Relations (QSVNR) as well as some properties of quadripartitioned single valued neutrosophic rough sets and its axiomatic characterizations. Under QSVN environment multicriteria decision making problems has been discussed in [16] and [18].

In this paper Section 2 deals with the basic definitions of QSVN set and the concept of K-algebras on single valued neutrosophic set. Section 3 discusses about K-algebras on QSVN, level subset of QSVN and also studies some of the results. Section 4 defines the homomorphism of quadripartitioned single valued neutrosophic K-algebras, characteristic and fully invariant K-subalgebras. Section 5 concludes the paper.

2| Preliminaries

This section deals with the basic definitions of QSVNS and K-algebra of single valued neutrosophic set that helps us to study the rest of the paper.

Definition 1. [19]. Let X be a non-empty set. A quadripartitioned neutrosophic set A over X characterizes each element x in X by a truth-membership function T_A, a contradiction membership function C_A, an ignorance – membership function U_A, and a falsity membership function F_A such that for each $x \in X, T_A, C_A, U_A, F_A \in \lfloor 0,1 \rfloor$ and $0 \leq T_A(x)+C_A(x)+U_A(x)+F_A(x) \leq 4$. When X is discrete A is represented as, $A=\sum_{i=1}^{n}\left\langle T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i)\right\rangle / x_i, x_i \in X.$

However, when the universe of discourse is continuous A is represented as $A=\int_X \left\langle T_A(x), C_A(x), U_A(x), F_A(x)\right\rangle / x, x \in X.$

Definition 2. [19]. Consider two QSVNS A and B over X. A is said to be contained in B, denoted by $A \subseteq B \text{ iff } T_A(x) \leq T_B(x), C_A(x) \leq C_B(x), U_A(x) \geq U_B(x) \text{ and } F_A(x) \geq F_B(x).$

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Definition 3. [19]. The complement of a QSVNS A is denoted by A^C and is defined as,

$$A^C = \sum_{i=1}^{n} \left\langle F_A(x_i), U_A(x_i), C_A(x_i), T_A(x_i) \right\rangle / x_i, x_i \in X,$$

$$T_{A^C}(x_i) = F_A(x_i), C_{A^C}(x_i) = U_A(x_i), U_{A^C}(x_i) = C_A(x_i) \text{ and } F_{A^C}(x_i) = T_A(x_i), x_i \in X.$$

Definition 4. [19]. The union of two QSVNS A and B is denoted by $A \cup B$ and is defined as,

$$A \cup B = \sum_{i=1}^{n} \left\langle T_A(x_i) \vee T_B(x_i), C_A(x_i) \vee C_B(x_i), U_A(x_i) \wedge U_B(x_i), F_A(x_i) \wedge F_B(x_i) \right\rangle / x_i, x_i \in X.$$

Definition 5. [19]. The intersection of two QSVNS A and B is denoted by $A \cap B$ and is defined as,

$$A \cap B = \sum_{i=1}^{n} \left\langle T_A(x_i) \wedge T_B(x_i), C_A(x_i) \wedge C_B(x_i), U_A(x_i) \vee U_B(x_i), F_A(x_i) \vee F_B(x_i) \right\rangle / x_i, x_i \in X.$$

Definition 6. [12]. Let $(G, \cdot, \Theta, e)$ be a group in which each non-identity element is not of order 2. Then a K-algebra is a structure $\mathcal{K} = (G, \cdot, \Theta, e)$ on a group G in which induced binary operation $\Theta: G \times G \to G$ is defined $b\, \Theta(x,y) = x\Theta y = xy^{-1}y$ and satisfies the following axioms:

$$(x\Theta y)\Theta(x\Theta z) = \left(x\Theta\left((e\Theta z)\Theta(e\Theta y)\right)\right)\Theta x,$$

$$x\Theta(x\Theta y) = \left(x\Theta(e\Theta y)\right)\Theta x,$$

$$x\Theta x = e,$$

$$x\Theta e = x,$$

$e\Theta x = x^{-1}$, for all $x, y, z \in G$.

Definition 7. [8]. A single-valued neutrosophic set $A = (T_A, I_A, F_A)$ in a K-algebra $\mathcal{K}$ is called a single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if it satisfies the following conditions:

$$T_A(s\Theta t) \geq \min\{T_A(s), T_A(t)\},$$

$$I_A(s\Theta t) \geq \min\{I_A(s), I_A(t)\},$$

$F_A(s\Theta t) \leq \max\{F_A(s), F_A(t)\}$, for all $s, t \in G$.

Note that $T_A(e) \geq T_A(s), I_A(e) \geq I_A(s), F_A(e) \leq F_A(s), \textit{for all } s \in G$.

3| Quadripartitioned Single Valued Neutrosophic K-Algebras

Definition 8. A quadripartitioned single valued neutrosophic set $X = (T_X, C_X, U_X, F_X)$ in a K-algebra $\mathcal{K}$ is called a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}$ if it satisfies the following conditions:

$T_X(e) \geq T_X(u), C_X(e) \geq C_X(u), U_X(e) \leq U_X(u), F_X(e) \leq F_X(u)$ for all $u \in G$.

$$T_X(u \,\Theta\, v) \geq \min\left\{ T_X(u), T_X(v) \right\},$$

$$C_X(u \,\Theta\, v) \geq \min\left\{ C_X(u), C_X(v) \right\},$$

$$U_X(u \,\Theta\, v) \leq \max\left\{ U_X(u), U_X(v) \right\},$$

$$F_X(u \,\Theta\, v) \leq \max\left\{ F_X(u), F_X(v) \right\} \text{ for all } u, v \in G.$$

Example 1. Let $G=\left\{ e, g, g^2, g^3, g^4 \right\}$ is the cyclic group of order 5 in a K-algebra $\mathcal{K}= (G, \cdot\ , \Theta, e)$. The Cayley's table for Θ is given as follows.

We define a quadripartitioned single valued neutrosophic set $X=(T_X, C_X, U_X, F_X)$ in K-algebra as follows:

$T_X(e)=0.5,$ $C_X(e)=0.7,$ $U_X(e)=0.3,$ $F_X(e)=0.5$
$T_X(u)=0.2,$ $C_X(u)=0.4,$ $U_X(u)=0.5,$ $F_X(u)=0.8$

for all $u \neq e \in G$. Clearly, it shows that $X=(T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-algebras of $\mathcal{K}$.

Proposition 1. If $X=(T_X, C_X, U_X, F_X)$ denotes a quadripartitioned single valued neutrosophic K-algebras of $\mathcal{K}$ then,

a) $(\forall\ u, v \in G), (T_X(u\Theta v) = T_X(v) \Rightarrow T_X(u) = T_X(e))$

$(\forall\ u, v \in G), \left(T_X(u) = T_X(e) \Rightarrow T_X(u\Theta v) \geq T_X(v)\right);$

b) $(\forall\ u, v \in G), (C_X(u\Theta v) = C_X(v) \Rightarrow C_X(u) = C_X(e))$

$(\forall\ u, v \in G), \left(C_X(u) = C_X(e) \Rightarrow C_X(u\Theta v) \geq C_X(v)\right);$

c) $(\forall\ u, v \in G), (U_X(u\Theta v) = U_X(v) \Rightarrow U_X(u) = U_X(e))$

$(\forall\ u, v \in G), \left(U_X(u) = U_X(e) \Rightarrow U_X(u\Theta v) \leq U_X(v)\right);$

d) $(\forall\ u, v \in G), (F_X(u\Theta v) = F_X(v) \Rightarrow F_X(u) = F_X(e))$

$(\forall\ u, v \in G), \left(F_X(u) = F_X(e) \Rightarrow F_X(u\Theta v) \geq F_X(v)\right);$

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Proof. We only prove (a) and (c). (b) and (d) proved in a similar way.

(a) First we assume that $T_X(u \odot v) = T_X(v)\ \forall\ u, v \in G$. Put $v = e$ and use (iii) of *Definition 6* we get $T_X(u) = T_X(u\Theta e) = T_X(e)$. Let for $u, v \in G$ be such that $T_X(u) = T_X(e)$ then$T_X(u\Theta v) \geq min\{T_X(u), T_X(v)\} = min\{T_X(e), T_X(v)\} = T_X(v)$.

Now to prove (c) consider that $U_X(u\Theta v) = U_X(v)\ \forall\ u, v \in$ G. Put $v = e$ and use (iii) of *Definition 6*, we have $U_X(u) = U_X(u\Theta e) = U_X(e)$. Let for $u, v \in G$ be such that $U_X(u) = U_X(e)$ then, $U_X(u\Theta v) \leq max\{U_X(u), U_X(v)\} = max\{U_X(e), U_X(v)\} = U_X(v)$. Hence the proof.

Definition 9. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of K and let $(\lambda, \mu, \vartheta, \xi) \in [0,1] \times [0,1] \times [0,1] \times [0,1]$ with $\lambda + \mu + \vartheta + \xi \leq 4$. Then the sets,

$$X_{(\lambda,\mu,\vartheta,\xi)} = \{u \in G |\ T_X(u) \geq \lambda,\ C_X(u) \geq \mu,\ U_X(u) \leq \vartheta,\ F_X(u) \leq \xi\},$$

$$(\lambda, \mu, \vartheta, \xi)X_{(\lambda,\mu,\vartheta,\xi)} = U(T_X, \lambda) \cap U'(C_X, \mu) \cap L(U_X, \vartheta) \cap L'(F_X, \xi)$$

are called $(\lambda, \mu, \vartheta, \xi)$ level subsets of quadripartitioned single valued neutrosophic set X.

And also the set $X_{(\lambda,\mu,\vartheta,\xi)} = \{u \in G |\ T_X(u) > \lambda,\ C_X(u) > \mu,\ U_X(u) < \vartheta,\ F_X(u) < \xi\}$ is known as strong level subset of X.

Note. The set of all $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ is known as image of $X = (T_X, C_X, U_X, F_X)$.

Proposition 2. If $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-algebra of K then the level subsets,

$$U(T_X, \lambda) = \{u \in G |\ T_X(u) \geq \lambda\}, U'(C_X, \mu) = \{u \in G |\ C_X(u) \geq \mu\},$$

$$L(U_X, \vartheta) = \{u \in G |\ U_X(u) \leq \vartheta\}, L'(F_X, \xi) = \{u \in G |\ F_X(u) \leq \xi\}$$

are K-subalgebras of K for every $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X) \subseteq [0,1]$

where $Im(T_X), Im(C_X), Im(U_X)$ *and* $Im(F_X)$ are sets of values $T(X), C(X), U(X)$ *and* $F(X)$, respectively.

Proof. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of K and$(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ be such that $U(T_X, \lambda) \neq \varnothing$, $U'(C_X, \mu) \neq \varnothing$, $L(U_X, \vartheta) \neq \varnothing$ *and* $L'(F_X, \xi) \neq \varnothing$. We have to show that U, U', L *and* L' are level K-subalgebras. Let for $u, v \in U(T_X, \lambda)$, $T_X(u) \geq \lambda$ *and* $T_X(v) \geq \lambda$. Then from *Definition 8* we get $T_X(u\Theta v) \geq min\{\ T_X(u),\ T_X(v)\} \geq \lambda$. It shows that $u\Theta v \in U(T_X, \lambda)$. Hence $U(T_X, \lambda)$ is a level K-subalgebra of K. Similarly, we can prove for $U'(C_X, \mu), L(U_X, \vartheta)$ *and* $L'(F_X, \xi)$.

Theorem 1. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of K. Then $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K if and only if $X_{(\lambda,\mu,\vartheta,\xi)}$ is a K-subalgebra of K for every $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ with $\lambda + \mu + \vartheta + \xi \leq 4$.

Proof. First assume tha$X_{(\lambda,\mu,\vartheta,\xi)}t$ is a K-subalgebra of K. If the conditions in *Definition 8* fail, then there exist $s, t \in G$ such that,

$$T_X(s\Theta t) < \min\{T_X(s), T_X(t)\},$$

$$C_X(s\Theta t) < \min\{C_X(s), C_X(t)\},$$

$$U_X(s\Theta t) > \max\{U_X(s), U_X(t)\},$$

$$F_X(s\Theta t) > \max\{F_X(s), F_X(t)\}.$$

Now let $\lambda_1 = \frac{1}{2}(T_X(s\Theta t) + min\{T_X(s), T_X(t)\})$, $\mu_1 = \frac{1}{2}(C_X(s\Theta t) + min\{C_X(s), C_X(t)\})$,

$\vartheta_1 = \frac{1}{2}(U_X(s\Theta t) + max\{U_X(s), U_X(t)\})$, $\xi_1 = \frac{1}{2}(F_X(s\Theta t) + max\{F_X(s), F_X(t)\})$.
Now we have,

$$T_X(s\Theta t) < \lambda_1 < \min\{T_X(s), T_X(t)\},$$

$$C_X(s\Theta t) < \mu_1 < \min\{C_X(s), C_X(t)\},$$

$$U_X(s\Theta t) > \vartheta_1 > \max\{U_X(s), U_X(t)\},$$

$$F_X(s\Theta t) > \xi_1 > \max\{F_X(s), F_X(t)\}.$$

This implies that $s, t \in X_{(\lambda,\mu,\vartheta,\xi)}$ *and* $s\Theta t \notin X_{(\lambda,\mu,\vartheta,\xi)}$ which is a contradiction. This proves that the conditions of *Definition 8* is true. Hence $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K.

Now assume that $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic K-subalgebra of K. Let for $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ with $\lambda + \mu + \vartheta + \xi \leq 4$ such that $X_{(\lambda,\mu,\vartheta,\xi)} \neq \emptyset$. Let $u, v \in X_{(\lambda,\mu,\vartheta,\xi)}$ be such that,

$$T_X(u) \geq \lambda, T_X(v) \geq \lambda',$$

$$C_X(u) \geq \mu, C_X(v) \geq \mu',$$

$$U_X(u) \leq \vartheta, U_X(v) \leq \vartheta',$$

$$F_X(u) \leq \xi, F_X(v) \leq \xi'.$$

Now assume that $\lambda \leq \lambda', \mu \leq \mu', \vartheta \geq \vartheta'$ *and* $\xi \geq \xi'$. It follows from *Definition 8* that,

$$T_X(u\Theta v) \geq \lambda = \min\{T_X(u), T_X(v)\},$$

$$C_X(u\Theta v) \geq \mu = \min\{C_X(u), C_X(v)\},$$

$$U_X(u\Theta v) \leq \vartheta = \max\{U_X(u), U_X(v)\},$$

$$F_X(u\Theta v) \leq \xi = \max\{F_X(u), F_X(v)\}.$$

This shows that $u\Theta v \in X_{(\lambda,\mu,\vartheta,\xi)}$. Hence $X_{(\lambda,\mu,\vartheta,\xi)}$ is a K-subalgebra of K.

Theorem 2. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic K-subalgebra and $(\lambda_1, \mu_1, \vartheta_1, \xi_1), (\lambda_2, \mu_2, \vartheta_2, \xi_2) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ with $\lambda_i + \mu_i + \vartheta_i + \xi_i \leq 4$ for $i = 1,2$. Then $X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)} = X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}$ if $(\lambda_1, \mu_1, \vartheta_1, \xi_1) = (\lambda_2, \mu_2, \vartheta_2, \xi_2)$.

Proof. When $(\lambda_1,\mu_1,\vartheta_1,\xi_1)=(\lambda_2,\mu_2,\vartheta_2,\xi_2)$ then the result is obvious for $X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)}=X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}$. Conversely assume that $X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)}=X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}$. Since $(\lambda_1,\mu_1,\vartheta_1,\xi_1)\in Im(T_X)\times Im(C_X)\times Im(U_X)\times Im(F_X)$ there exists $u\in G$ such that $T_X(u)=\lambda_1, C_X(u)=\mu_1, U_X(u)=\vartheta_1$ *and* $F_X(u)=\xi_1$. This implies that $u\in X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)}=X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}$. Hence $\lambda_1=T_X(u)\geq\lambda_2, \mu_1=C_X(u)\geq\mu_2, \vartheta_1=U_X(u)\leq\vartheta_2$ *and* $\xi_1=F_X(u)\leq\xi_2$. Also $(\lambda_2,\mu_2,\vartheta_2,\xi_2)\in Im(T_X)\times Im(C_X)\times Im(U_X)\times Im(F_X)$ there exists $v\in G$ such that $T_X(v)=\lambda_2$, $C_X(v)=\mu_2, U_X(v)=\vartheta_2$ *and* $F_X(v)=\xi_2$. This implies that $v\in X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}=X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)}$. Hence $\lambda_2=T_X(v)\geq\lambda_1, \mu_2=C_X(v)\geq\mu_1, \vartheta_2=U_X(v)\leq\vartheta_1$ *and* $\xi_2=F_X(v)\leq\xi_1$. Hence $(\lambda_1,\mu_1,\vartheta_1,\xi_1)=(\lambda_2,\mu_2,\vartheta_2,\xi_2)$.

Theorem 3. Let I be a K-subalgebra of K-algebra $\mathcal{K}$. Then there exists a quadripartitioned single valued neutrosophic K-subalgebra $X=\left(T_X,C_X,U_X,F_X\right)$ of K-algebra $\mathcal{K}$ such that $X=\left(T_X,C_X,U_X,F_X\right)=I$ for some $\lambda,\mu\in(0,1]$ *and* $\vartheta,\xi\in[0,1)$.

Proof. Let $X=(T_X,C_X,U_X,F_X)$ be a quadripartitioned single valued neutrosophic set in K-algebra $\mathcal{K}$ given by,

$$T_X(u)=\begin{cases}\lambda\in(0,1], & \text{if } u\in I\\ 0, & \text{otherwise}\end{cases}$$

$$C_X(u)=\begin{cases}\mu\in(0,1], & \text{if } u\in I\\ 0, & \text{otherwise}\end{cases}$$

$$U_X(u)=\begin{cases}\vartheta\in[0,1), & \text{if } u\in I\\ 0, & \text{otherwise}\end{cases}$$

$$F_X(u)=\begin{cases}\xi\in[0,1), & \text{if } u\in I\\ 0, & \text{otherwise}\end{cases}.$$

Let $u,v\in G$. If $u,v\in I$, then $u\Theta v\in I$ and so,

$$T_X(u\Theta v)\geq\min\{T_X(u),T_X(v)\},$$

$$C_X(u\Theta v)\geq\min\{C_X(u),C_X(v)\},$$

$$U_X(u\Theta v)\leq\max\{U_X(u),U_X(v)\},$$

$$F_X(u\Theta v)\leq\max\{F_X(u),F_X(v)\}.$$

Suppose $u\notin I$ *or* $v\notin I$ then,

$$T_X(u)=0 \text{ or } T_X(v), C_X(u)=0 \text{ or } C_X(v), U_X(u)=0 \text{ or } U_X(v) \text{ and } F_X(u)=0 \text{ or } F_X(v).$$

It implies that,

$$T_X(u\Theta v)\geq\min\{T_X(u),T_X(v)\},$$

$$C_X(u\Theta v)\geq\min\{C_X(u),C_X(v)\},$$

$$U_X(u\Theta v)\leq\max\{U_X(u),U_X(v)\},$$

$$F_X(u\Theta v)\leq\max\{F_X(u),F_X(v)\}.$$

Hence $X=(T_X,C_X,U_X,F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}$.

Consequently $X_{(\lambda,\mu,\vartheta,\xi)} = I$

Theorem 4. Let $\bar{K}$ be a K-algebra. Let a chain of K-subalgebras: $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = G$. Then the level K-subalgebras of the quadripartitioned single valued neutrosophic K-subalgebra remains same as the K-subalgebras of this chain.

Proof. Let $\{\lambda_i|i = 0,1,\dots,n\},\{\mu_i|i = 0,1,\dots,n\}$ be finite decreasing sequences and $\{\vartheta_i|i = 0,1,\dots,n\}$, $\{\xi_i|i = 0,1,\dots,n\}$ be finite increasing sequences in $[0,1]$ such that $\lambda_k + \mu_k + \vartheta_k + \xi_k \leq 4$ for $k = 0,1,2,\dots,n$. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in $\bar{K}$ defined by $T_X(X_0) = \lambda_0$, $C_X(X_0) = \mu_0$, $U_X(X_0) = \vartheta_0$ *and* $F_X(X_0) = \xi_0$,

$T_X(X_i \backslash X_{i-1}) = \lambda_i$, $C_X(X_i \backslash X_{i-1}) = \mu_i$, $U_X(X_i \backslash X_{i-1}) = \vartheta_i$ *and* $F_X(X_i \backslash X_{i-1}) = \xi_i$ *for* $0 < i \leq n$.

We have $t\, u\Theta v \in X_{i-1} o$ prove that $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\bar{K}$. Let $u, v \in G$. If $u, v \in X_i \backslash X_{i-1}$ then it implies that $T_X(u) = \lambda_i = T_X(v)$, $C_X(u) = \mu_i = C_X(v)$, $U_X(u) = \vartheta_i = U_X(v)$ and $F_X(u) = \xi_i = F_X(v)$. Since each X_i is a K-subalgebra, we get $u\Theta v \in X_i$. So that either $u\Theta v \in X_i \backslash X_{i-1}$ or. In any of the above case it follows that,

$$T_X(u\Theta v) \geq \lambda_i = \min\{T_X(u), T_X(v)\},$$

$$C_X(u\Theta v) \geq \mu_i = \min\{C_X(u), C_X(v)\},$$

$$U_X(u\Theta v) \leq \vartheta_i = \max\{U_X(u), U_X(v)\},$$

$$F_X(u\Theta v) \leq \xi_i = \max\{F_X(u), F_X(v)\}.$$

For $k > l$ if $u \in X_k \backslash X_{k-1}$ and $v \in X_l \backslash X_{l-1}$ then,

$$T_X(u) = \lambda_k, T_X(v) = \lambda_l,$$

$$C_X(u) = \mu_k, C_X(v) = \mu_l,$$

$$U_X(u) = \vartheta_k, U_X(v) = \vartheta_l,$$

$$F_X(u) = \xi_k, F_X(v) = \xi_l,$$

and $u\Theta v \in X_k$ because X_k is a K-subalgebra and $X_l \subset X_k$. It follows that,

$$T_X(u\Theta v) \geq \lambda_k = \min\{T_X(u), T_X(v)\},$$

$$C_X(u\Theta v) \geq \mu_k = \min\{C_X(u), C_X(v)\},$$

$$U_X(u\Theta v) \leq \vartheta_k = \max\{U_X(u), U_X(v)\},$$

$$F_X(u\Theta v) \leq \xi_k = \max\{F_X(u), F_X(v)\}.$$

Hence $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\bar{K}$ and all its non-empty level subsets are level K-subalgebras of $\bar{K}$. Since $Im(T_X) = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$, $Im(C_X) = \{\mu_0, \mu_1, \dots, \mu_n\}$, $Im(U_X) = \{\vartheta_0, \vartheta_1, \dots, \vartheta_n\}$ *and* $Im(F_X) = \{\xi_0, \xi_1, \dots, \xi_n\}$. Therefore, the level K-subalgebras of $X = (T_X, C_X, U_X, F_X)$ are given by the chain of K-subalgebras:

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$$U(T_X, \lambda_0) \subset U(T_X, \lambda_1) \subset \cdots \subset U(T_X, \lambda_n) = G,$$

$$U'(C_X, \mu_0) \subset U'(C_X, \mu_1) \subset \cdots \subset U'(C_X, \mu_n) = G,$$

$$L(U_X, \vartheta_0) \subset L(U_X, \vartheta_1) \subset \cdots \subset L(U_X, \vartheta_n) = G,$$

$$L'(F_X, \xi_0) \subset L'(F_X, \xi_1) \subset \cdots \subset L'(F_X, \xi_n) = G,$$

respectively. Indeed,

$$U(T_X, \lambda_0) = \{u \in G |\ T_X(u) \geq \lambda_0\} = X_0,$$

$$U'(C_X, \mu_0) = \{u \in G |\ C_X(u) \geq \mu_0\} = X_0,$$

$$L(U_X, \vartheta_0) = \{u \in G |\ U_X(u) \leq \vartheta_0\} = X_0,$$

$$L'(F_X, \xi_0) = \{u \in G |\ F_X(u) \leq \xi_0\} = X_0.$$

Now we have to prove that,

$U(T_X, \lambda_i) = X_i, U'(C_X, \mu_i) = X_i, L(U_X, \vartheta_i) = X_i$ *and* $L'(F_X, \xi_i) = X_i$ *for* $0 < i \leq n$. Clearly $X_i \subseteq U(T_X, \lambda_i), X_i \subseteq U'(C_X, \mu_i), X_i \subseteq L(U_X, \vartheta_i)$ *and* $X_i \subseteq L'(F_X, \xi_i)$. If $u \in U(T_X, \lambda_i)$ then $T_X(u) \geq \lambda_i$ and so $u \notin A_k$ *for* $k > i$. Hence $T_X(u) \in \{\lambda_0, \lambda_1, \dots, \lambda_i\}$ which shows that $u \in X_k$ *for* $k \leq i$, since $X_k \subseteq X_{i.}$ It follows that $u \in X_i$. Consequently $U(T_X, \lambda_i) = X_i$ for some $0 < i \leq n$. Similarly, it is proved for $U'(C_X, \mu_i) = X_i$. Now if $v \in L(U_X, \vartheta_i)$ then $U_X(v) \leq \vartheta_i$ and so $v \notin X_k$ *for some* $i \leq k$. Thus $U_X(u) \in \{\vartheta_0, \vartheta_1, \dots, \vartheta_i\}$ which shows that $u \in X_l$ *for some* $l \leq i$, since $X_l \subseteq X_{i.}$ It follows that $v \in X_i$. Consequently, $L(U_X, \vartheta_i) = X_i$ for some $0 < i \leq n$. Similarly, it is proved for $L'(F_X, \xi_i) = X_i$. Hence the proof.

4 | Homomorphism of Quadripartitioned Single Valued Neutrosophic K-Algebras

Definition 10. Consider two K-algebras $\mathcal{K}_1 = (G_1, \cdot, \mathcal{O}, e_1)$ *and* $\mathcal{K}_2 = (G_2, \cdot, \mathcal{O}, e_2)$ and f be a function from $\mathcal{K}_1$into $\mathcal{K}_2$. If $Y = (T_Y, C_Y, U_Y, F_Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_2$, then the preimage of $Y = (T_Y, C_Y, U_Y, F_Y)$ under f is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_1$ defined by,

$$f^{-1}(T_Y)(u) = T_Y\big(f(u)\big), f^{-1}(C_Y)(u) = C_Y\big(f(u)\big),$$

$$f^{-1}(U_Y)(u) = U_Y\big(f(u)\big), f^{-1}(F_Y)(u) = F_Y\big(f(u)\big),$$

for all $u \in G$.

Definition 11. A quadripartitioned single valued neutrosophic K-subalgebra $X = (T_X, C_X, U_X, F_X)$ of a K-algebra $\mathcal{K}$ is called characteristic if $T_X\big(f(u)\big) = T_X(u), C_X\big(f(u)\big) = C_X(u), U_X\big(f(u)\big) = U_X(u)$ *and* $F_X\big(f(u)\big) = F_X(u)$ for all $u \in G$ and $f \in Aut(K)$.

Definition 12. A K-subalgebra U of a K-algebra $\mathcal{K}$ is said to be fully invariant if $f(U) \subseteq U$ for all $f \in End(K)$ where $End(K)$ is the set of all endomorphisms of a K-algebra $\mathcal{K}$. A quadripartitioned single valued neutrosophic K-subalgebra $X = (T_X, C_X, U_X, F_X)$ of a K-algebra $\mathcal{K}$ is called fully invariant if $T_X\big(f(u)\big) \leq T_X(u), C_X\big(f(u)\big) \leq C_X(u), U_X\big(f(u)\big) \geq U_X(u)$ *and* $F_X\big(f(u)\big) \geq F_X(u)$ for all $u \in G$ and $f \in End(K)$.

Definition 13. Let $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ and $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ be two quadripartitioned single valued neutrosophic K-subalgebras of $\mathcal{K}$. Then $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ is said to be the same type of $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ if there exists $f \in Aut(K)$ such that $X_1 = X_2 \circ f$ i.e.,$T_{X_1}(u) = T_{X_2}\big(f(u)\big), C_{X_1}(u) = C_{X_2}\big(f(u)\big), U_{X_1}(u) = U_{X_2}\big(f(u)\big)$ *and* $F_{X_1}(u) = F_{X_2}\big(f(u)\big)$ for all $u \in G$.

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Theorem 5. Let $f: \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $Y = (T_Y, C_Y, U_Y, F_Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_2$, then $f^{-1}(Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Proof. It is obvious that,

$$f^{-1}(T_Y)(e) \geq f^{-1}(T_Y)(u), f^{-1}(C_Y)(e) \geq f^{-1}(C_Y)(u),$$

$$f^{-1}(U_Y)(e) \leq f^{-1}(U_Y)(u), f^{-1}(F_Y)(e) \leq f^{-1}(F_Y)(u),$$

for all $u \in G_1$. *Let* $u, v \in G_1$ then,

$$f^{-1}(T_Y)(u\Theta v) = T_Y(f(u\Theta v)),$$

$$f^{-1}(T_Y)(u\Theta v) = T_Y(f(u)\Theta f(v)),$$

$$f^{-1}(T_Y)(u\Theta v) \geq \min\Big\{T_Y\big(f(u)\big), T_Y\big(f(v)\big)\Big\},$$

$$f^{-1}(T_Y)(u\Theta v) \geq \min\Big\{f^{-1}(T_Y)(u), f^{-1}(T_Y)(v)\Big\};$$

$$f^{-1}(C_Y)(u\Theta v) = C_Y(f(u\Theta v)),$$

$$f^{-1}(C_Y)(u\Theta v) = C_Y(f(u)\Theta f(v)),$$

$$f^{-1}(C_Y)(u\Theta v) \geq \min\Big\{C_Y\big(f(u)\big), C_Y\big(f(v)\big)\Big\},$$

$$f^{-1}(C_Y)(u\Theta v) \geq \min\Big\{f^{-1}(C_Y)(u), f^{-1}(C_Y)(v)\Big\};$$

$$f^{-1}(U_Y)(u\Theta v) = U_Y(f(u\Theta v)),$$

$$f^{-1}(U_Y)(u\Theta v) = U_Y(f(u)\Theta f(v)),$$

$$f^{-1}(U_Y)(u\Theta v) \leq \max\Big\{U_Y\big(f(u)\big), U_Y\big(f(v)\big)\Big\},$$

$$f^{-1}(U_Y)(u\Theta v) \leq \max\Big\{f^{-1}(U_Y)(u), f^{-1}(U_Y)(v)\Big\};$$

$$f^{-1}(F_Y)(u\Theta v) = F_Y(f(u\Theta v)),$$

$$f^{-1}(F_Y)(u\Theta v) = F_Y(f(u)\Theta f(v)),$$

$$f^{-1}(F_Y)(u\Theta v) \leq \max\Big\{F_Y\big(f(u)\big), F_Y\big(f(v)\big)\Big\},$$

$$f^{-1}(F_Y)(u\Theta v) \leq \max\Big\{f^{-1}(F_Y)(u), f^{-1}(F_Y)(v)\Big\}.$$

Hence $f^{-1}(Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Theorem 6. Let $f: \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $Y = (T_Y, C_Y, U_Y, F_Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_2$ and $X = \left(T_X, C_{X,} U_X, F_X\right)$ is the preimage of Y under f. Then X is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Proof. It is obvious that $T_X(e) \geq T_X(u), C_X(e) \geq C_X(u), U_X(e) \leq U_X(u)$ *and* $F_X(e) \leq F_X(u)$

for all $u \in G_1$. Now for any $u, v \in G_1$,

$$\mathrm{T_X(u \odot v) = T_Y(f(u \odot v))},$$

$$\mathrm{T_X(u \odot v) = T_Y(f(u) \odot f(v))},$$

$$\mathrm{T_X(u \odot v) \geq \min\left\{T_Y\left(f(u)\right), T_Y\left(f(v)\right)\right\}},$$

$$\mathrm{T_X(u \odot v) \geq \min\{T_X(u), T_X(v)\}};$$

$$\mathrm{C_X(u \odot v) = C_Y(f(u \odot v))},$$

$$\mathrm{C_X(u \odot v) = C_Y(f(u) \odot f(v))},$$

$$\mathrm{C_X(u \odot v) \geq \min\left\{C_Y\left(f(u)\right), C_Y\left(f(v)\right)\right\}},$$

$$\mathrm{C_X(u \odot v) \geq \min\{C_X(u), C_X(v)\}};$$

$$\mathrm{U_X(u \odot v) = U_Y(f(u \odot v))},$$

$$\mathrm{U_X(u \odot v) = U_Y(f(u) \odot f(v))},$$

$$\mathrm{U_X(u \odot v) \leq \max\left\{U_Y\left(f(u)\right), U_Y\left(f(v)\right)\right\}},$$

$$\mathrm{U_X(u \odot v) \leq \max\{U_X(u), U_X(v)\}};$$

$$\mathrm{F_X(u \odot v) = F_Y(f(u \odot v))},$$

$$\mathrm{F_X(u \odot v) = F_Y(f(u) \odot f(v))},$$

$$\mathrm{F_X(u \odot v) \leq \max\left\{F_Y\left(f(u)\right), F_Y\left(f(v)\right)\right\}},$$

$$\mathrm{F_X(u \odot v) \leq \max\{F_X(u), F_X(v)\}}.$$

Hence X is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Definition 14. Let f be a mapping from $\mathcal{K}_1$into $\mathcal{K}_2$ i.e., $f: \mathcal{K}_1 \to \mathcal{K}_2$ of K-algebras and let $X = (T_X, C_{X,} U_X, F_X)$ be a quadripartitioned single valued neutrosophic set of $\mathcal{K}_2$. The map $X = (T_X, C_{X,} U_X, F_X)$ is called the preimage of X under f if $T_X^f(u) = T_X\left(f(u)\right), C_X^f(u) = C_X\left(f(u)\right), U_X^f(u) = U_X\left(f(u)\right)$ *and* $F_X^f(u) = F_X\left(f(u)\right)$ for all $u \in G_1$.

Theorem 7. Let $f: K_1 \to K_2$ be an epimorphism of K-algebras. Then $X^f = (T_X^f, C_X^f, U_X^f, F_X^f)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_1 if and only if $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2.

Proof. Let $f: K_1 \to K_2$ be an epimorphism of K-algebras. First assume that $X^f = (T_X^f, C_X^f, U_X^f, F_X^f)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_1. Then we have to prove that $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2. Since there exists $u \in G_1$ such that $v = f(u)$ for any $v \in G_2$:

$$T_X(v) = T_X\big(f(u)\big) = T_X^{f(u)} \leq T_X^{f(e_1)} = T_X\big(f(e_1)\big) = T_X(e_2),$$

$$C_X(v) = C_X\big(f(u)\big) = C_X^{f(u)} \leq C_X^{f(e_1)} = C_X\big(f(e_1)\big) = C_X(e_2),$$

$$U_X(v) = U_X\big(f(u)\big) = U_X^{f(u)} \geq U_X^{f(e_1)} = U_X\big(f(e_1)\big) = U_X(e_2),$$

$$F_X(v) = F_X\big(f(u)\big) = F_X^{f(u)} \geq F_X^{f(e_1)} = F_X\big(f(e_1)\big) = F_X(e_2).$$

For any $u, v \in G_2, s, t \in G_1$ such that $u = f(s)$ and $v = f(t)$. It follows that:

$$T_X(u\Theta v) = T_X(f(s\Theta t)),$$

$$T_X(u\Theta v) = T_X^f(s\Theta t),$$

$$T_X(u\Theta v) \geq \min\{T_X^f(s), T_X^f(t)\},$$

$$T_X(u\Theta v) \geq \min\{T_X\big(f(s)\big), T_X\big(f(t)\big)\},$$

$$T_X(u\Theta v) \geq \min\{T_X(u), T_X(v)\};$$

$$C_X(u\Theta v) = C_X\big(f(s\Theta t)\big),$$

$$C_X(u\Theta v) = C_X^f(s\Theta t),$$

$$C_X(u\Theta v) \geq \min\{C_X^f(s), C_X^f(t)\},$$

$$C_X(u\Theta v) \geq \min\{C_X\big(f(s)\big), C_X\big(f(t)\big)\},$$

$$C_X(u\Theta v) \geq \min\{C_X(u), C_X(v)\};$$

$$U_X(u\Theta v) = U_X\big(f(s\Theta t)\big),$$

$$U_X(u\Theta v) = U_X^f(s\Theta t),$$

$$U_X(u\Theta v) \leq \max\{U_X^f(s), U_X^f(t)\},$$

$$U_X(u\Theta v) \leq \max\{U_X\big(f(s)\big), U_X\big(f(t)\big)\},$$

$$U_X(u\Theta v) \leq \max\{U_X(u), U_X(v)\};$$

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$$F_X(u\odot v) = F_X\big(f(s\Theta t)\big),$$

$$F_X(u\Theta v) = F_X^f(s\Theta t),$$

$$F_X(u\Theta v) \leq \max\{F_X^f(s), F_X^f(t)\},$$

$$F_X(u\odot v) \leq \max\{F_X\big(f(s)\big), F_X\big(f(t)\big)\},$$

$$F_X(u\odot v) \leq \max\{F_X(u), F_X(v)\}.$$

Hence $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2. Conversely, assume that $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2. Then we have to prove that $X^f = (T_X^f, C_X^f, U_X^f, F_X^f)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_1. For any $u \in G_1$ we have:

$$T_X^f(e_1) = T_X\big(f(e_1)\big) = T_X(e_2) \geq T_X\big(f(u)\big) = T_X^f(u),$$

$$C_X^f(e_1) = C_X\big(f(e_1)\big) = C_X(e_2) \geq C_X\big(f(u)\big) = C_X^f(u),$$

$$U_X^f(e_1) = U_X\big(f(e_1)\big) = U_X(e_2) \leq U_X\big(f(u)\big) = U_X^f(u),$$

$$F_X^f(e_1) = F_X\big(f(e_1)\big) = F_X(e_2) \leq F_X\big(f(u)\big) = F_X^f(u).$$

Since X is a quadripartitioned single valued neutrosophic K-subalgebra of K_2 and for any $u, v \in G_1$,

$$T_X^f(u\Theta v) = T_X\big(f(u\Theta v)\big),$$

$$T_X^f(u\Theta v) = T_X\big(f(u)\Theta f(v)\big),$$

$$T_X^f(u\Theta v) \geq \min\{T_X\big(f(u)\big), T_X\big(f(v)\big)\},$$

$$T_X^f(u\Theta v) \geq \min\{T_X^f(u), T_X^f(v)\};$$

$$C_X^f(u\Theta v) = C_X\big(f(u\Theta v)\big),$$

$$C_X^f(u\Theta v) = C_X\big(f(u)\Theta f(v)\big),$$

$$C_X^f(u\Theta v) \geq \min\{C_X\big(f(u)\big), C_X\big(f(v)\big)\},$$

$$C_X^f(u\Theta v) \geq \min\{C_X^f(u), C_X^f(v)\};$$

$$U_X^f(u\Theta v) = U_X(f(u\Theta v)),$$

$$U_X^f(u\Theta v) = U_X\big(f(u)\Theta f(v)\big),$$

$$U_X^f(u\Theta v) \leq \max\{U_X(f(u)), U_X(f(v))\},$$

$$U_X^f(u \odot v) \le \max\{U_X^f(u), U_X^f(v)\};$$

$$F_X^f(u \odot v) = F_X(f(u \odot v)),$$

$$F_X^f(u \odot v) = F_X(f(u) \odot f(v)),$$

$$F_X^f(u \odot v) \le \max\{F_X(f(u)), F_X(f(v))\},$$

$$F_X^f(u \odot v) \le \max\{F_X^f(u), F_X^f(v)\}.$$

Hence $X^f = (T_X^f, C_X^f, U_X^f, F_X^f)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathcal{K}_1$.

Theorem 8. Let $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ and $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ be two quadripartitioned single valued neutrosophic K-subalgebras of $\mathcal{K}$. Then a quadripartitioned single valued neutrosophic K-subalgebra $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ is of the same type of quadripartitioned single valued neutrosophic K-subalgebra $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ if and only if X_1 is isomorphic to X_2.

Proof. It is enough to prove only the necessary condition since sufficient condition holds trivially. Let $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ be quadripartitioned single valued neutrosophic K-subalgebra having same type of $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$. Then there exists $f \in Aut(K)$ such that $T_{X_1}(u) = T_{X_2}(f(u)), C_{X_1}(u) = C_{X_2}(f(u)), U_{X_1}(u) = U_{X_2}(f(u))$ *and* $F_{X_1}(u) = F_{X_2}(f(u))$ for all $u \in G$.

Let $g: X_1(K) \to X_2(K)$ be a mapping defined by $g(X_1(s)) = X_2(f(u))$ for all $u \in G$ i.e., $g(T_{X_1}(u)) = T_{X_2}(f(u)), g(C_{X_1}(u)) = C_{X_2}(f(u)), g(U_{X_1}(u)) = U_{X_2}(f(u))$ *and* $g(F_{X_1}(u)) = F_{X_2}(f(u))$ for all $u \in G$. g is surjective obviously. And if $g(T_{X_1}(u)) = g(T_{X_1}(v))$ for all $u, v \in G$ then $T_{X_2}(f(u)) = T_{X_2}(f(v))$ and we get $T_{X_1}(u) = T_{X_1}(v)$. Similarly we can prove for $C_{X_1}(u) = C_{X_1}(v), U_{X_1}(u) = U_{X_1}(v)$ *and* $F_{X_1}(u) = F_{X_1}(v)$.

Hence g is injective. Therefore g is a homomorphism such that for $u, v \in G$ we have:

$$g(T_{X_1}(u \odot v)) = T_{X_2}(f(u \odot v)) = T_{X_2}(f(u) \odot f(v)),$$

$$g(C_{X_1}(u \odot v)) = C_{X_2}(f(u \odot v)) = C_{X_2}(f(u) \odot f(v)),$$

$$g(U_{X_1}(u \odot v)) = U_{X_2}(f(u \odot v)) = U_{X_2}(f(u) \odot f(v)),$$

$$g(F_{X_1}(u \odot v)) = F_{X_2}(f(u \odot v)) = F_{X_2}(f(u) \odot f(v)).$$

Hence $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ is isomorphic to $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$.

5 | Conclusion

In recent years, a new branch of logical algebra known as K-algebra applied in fuzzy set, intuitionistic fuzzy set and single valued neutrosophic set which helps us to extend the concept to K-algebra on quadripartitioned single valued neutrosophic sets. Quadripartitioned single valued neutrosophic set has four components truth,contradiction,unknown,false which helps to deal the concept of indeterminacy effectively. In this paper we defined K-algebras on quadripartitioned single valued neutrosophic sets and studied some of the results. Further the homomorphism of quadripartitioned single valued neutrosophic K-algebras, characteristic and fully invariant K-subalgebras also discussed in detail.

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